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A duality theorem for homomorphisms between generalized Verma modules

By

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Introduction

Let K be a field of characteristic zero, $\mathfrak{g}$ a split semisimple Lie algebra over K, $\mathfrak{p}$ a parabolic subalgebra, and ε the half of the sum of roots whose root subspaces are contained in the nilpotent radical of $\mathfrak{p}$. Then -2ε gives a one dimensional $\mathfrak{p}$-module, which we denote by the same letter. For a finite dimensional simple $\mathfrak{p}$-module E, let E^* be its dual $\mathfrak{p}$-module. Put $M(E)=U(\mathfrak{g})\otimes_{U(\mathfrak{p})}E$. The following duality theorem is attributed to G. Zuckerman (cf. [1,(4.9)]).

Duality Theorem. *For a finite dimensional simple $\mathfrak{p}$-modules E and F, there is a natural isomorphism*

$$\mathrm{Hom}_{\mathfrak{g}}(M(E),M(F))\simeq \mathrm{Hom}_{\mathfrak{g}}(M(F^*\otimes(-2\varepsilon)),\ M(E^*\otimes(-2\varepsilon))).$$

In order to study the b-functions of semi-invariants and the generalized Verma modules [10], the author has come to need [1,(4.9)]. Since [1,(4.9)] seems difficult to understand correctly for non-experts, we give in this note a detailed proof, which follows a similar line as was indicated in [1,(4.9)], but is purely algebraic.

Convention. For an algebra A, an A-module means a left A-module, unless otherwise stated. Every vector space is considered over the base field K, and, Hom and $\otimes$ means Hom_K and $\otimes_K$. For a vector space V, V^* denotes its dual space, and $\langle\ \rangle$ the natural pairing of V and V^*. More generally, we sometimes denote the value of a (vector valued) function f at a point p by $\langle f,p\rangle$ or $\langle p,f\rangle$ for $f(p)$.

A Lie algebra character, say λ, of a Lie algebra $\mathfrak{g}$ gives a one dimensional $\mathfrak{g}$-module, which we shall denote by the same letter λ. We consider K as the trivial $\mathfrak{g}$-module, which is also denoted by 0 by the above convention.

When two objects are *naturally* isomorphic, we sometimes write $=$ for $\simeq$.

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The purpose of this section is to prove (1.7), which is used later in (3.7).

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