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# Busemann functions and positive eigenfunctions of Laplacian on noncompact symmetric spaces

By

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## Introduction

Let $X$ be a complete simply connected manifold of nonpositive sectional curvature. We can associate each geodesic ray $\gamma$ in $X$ with the following function $b(\gamma)$:

$$b(\gamma)(x)=\lim_{t\to\infty}\{d(x,\ \gamma(t))-t\} \qquad \text{for } x\in X, \tag{0.1}$$

where $d$ is the distance on $X$. This is called the Busemann function associated with $\gamma$ (which was defined in [7]) and is an important object in the study of nonpositively curved manifolds. It is a $C^2$ convex function and the inverse images $b(\gamma)^{-1}(t)$ ($t\in\mathbf{R}$) are called the horospheres ([13,17]). By investigating such objects, many results, for example, concerning co-finite discrete groups $\Gamma$ of isometries of $X$ and the structure of the ends of the quotient spaces $\Gamma\backslash X$ were obtained (e.g. [3,10]).

In this paper we point out that the Busemann function has other aspects which do not appear in its geometric definition in the case of symmetric spaces of noncompact type.

Let us consider the case where $X$ has constant sectional curvature $-1$. In this case, the functions $e^{-b(\gamma)(x)}$ are minimal positive harmonic functions as pointed out in [2]. We can show this fact by direct computation. On the other hand, the author computed the Busemann functions on the symmetric space $SO(n)\backslash SL(n,\mathbf{R})$ in ([15, 16]). The result is as follows. Let $P(n,\mathbf{R})$ be the set of all positive definite symmetric matrices with determinant 1. If we identify $SO(n)\backslash SL(n,\mathbf{R})$ with $P(n,\mathbf{R})$ in the usual manner, the Busemann function $b(\gamma)$ associated with the geodesic ray

$$\gamma(t)=\mathrm{diag}(e^{2t\alpha_1/\|\alpha\|},e^{2t\alpha_2/\|\alpha\|},\cdots,e^{2t\alpha_n/\|\alpha\|})$$

is

$$b(\gamma)(x)=\frac{n}{\|\alpha\|}\log\Bigl(\prod_{i=1}^{n-1}\triangle_i(x)^{\alpha_{i+1}-\alpha_i}\Bigr) \qquad \text{for } x\in P(n,\mathbf{R}), \tag{0.2}$$

where $\alpha=\mathrm{diag}(\alpha_1,\cdots,\alpha_n)$ is an element of the Lie algebra of $SL(n,\mathbf{R})$ such that $\alpha_1<\alpha_2<\cdots<\alpha_n$, $\|\alpha\|$ its norm with respect to the metric induced from the Killing form,

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