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The Atkin inner product for $\Gamma_0(N)$

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1. Introduction

Let $\mathfrak{H}$ be the complex upper half plane and $\mathcal{M}$ the $\mathbb{C}$-vector space of $SL_2(\mathbb{Z})$-invariant functions which are holomorphic on $\mathfrak{H}$ and meromorphic at $i\infty$. The space $\mathcal{M}$ can be identified with the polynomial ring $\mathbb{C}[j]$ via $j = j(\tau)$, where $j(\tau)$ is the elliptic modular invariant:

$$j(\tau) = \frac{1}{q} + 744 + 196884q + 21493760q^2 + 864299970q^3 + \dots, \quad q = e^{2\pi i\tau}.$$

On the space $\mathcal{M}$ act the Hecke operators $\{T_n\}_{n\in\mathbb{N}}$, and on $\mathbb{C}[j]$ too, through the above identification.

A.O.L Atkin defined an inner product $(\ , \ )$ on $\mathcal{M}$ by

$$(f \ , \ g) = \text{constant term of } \ f \cdot g\, E_2 \ \text{ as Laurent series in } \ q = e^{2\pi i\tau},$$

where $E_2(\tau)$ is the Eisenstein series of weight 2 for $SL_2(\mathbb{Z})$:

$$E_2(\tau) = 1 - 24 \sum_{m=1}^{\infty} \left(\sum_{d|m} d \right) q^m.$$

Atkin showed:

1. The Hecke operators T_n are self-adjoint with respect to this inner product;
$$(f|T_n, \ g) = (f \ , g|T_n), \qquad \forall f, \ g \in \mathcal{M}, \ \forall n \geq 1.$$
2. The inner product is non-degenerate and the associated orthogonal polynomials are connected to the j-invariants of supersingular elliptic curves. (For precise statement, see the article [5].)

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