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Prof. Yosida's proof of the Plancherel and the Bochner theorems for locally compact abelian groups

By

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Banach algebras were introduced by M. Nagumo [7] and K. Yosida [10] in 1936. I. M. Gelfand's celebrated paper [2] appeared five years later. The theory was immediately applied to the reconstruction of the theory of locally compact abelian groups. In the Soviet Union, Gelfand, D. A. Raikov, M. Krein and M. Neumark published short notes in Doklady Nauk in 1940–1944 but the details were shown to the public only after the war as a book [4]. Under the isolation caused by the war, similar works were done in France by H. Cartan and R. Godement, and in the United States by I. E. Segal independently. They also had to wait until the end of the war before they published their results (see Introductions in Cartan-Godement [1] and Segal [8]).

On the contrary, Japanese were able to read Doklady and publish their works. In 1944 Professor Yosida published two papers on this subject. In [12] he claimed that he improved Krein's proof [6] of the Plancherel theorem, and in [13] Raikov's proof [9] of the Bochner theorem. I think his proofs are the most natural ones but they did not attract due attention, probably because his proofs yet contained some gaps. I showed how to fill in the gaps in Comments of his Collected Papers [14] by quoting a book [5]. However, since I have again found a gap in the book, I like to fill it in here.

Let X be a locally compact abelian group with a Haar measure dx. In the theory of Gelfand et al. they use the group ring $\mathbf{B}_0 = L^1(X)$ with convolution as multiplication. By adding the unity 1 if X is not discrete, we obtain a commutative Banach algebra $\mathbf{B}$ with 1. The starting point was Gelfand-Raikov's observation [3] that the space $\operatorname{Spec}\mathbf{B}$ of maximal ideals of $\mathbf{B}$ is identified with the character group $\hat{X}$ except for the point $M_\infty = L^1(X)$ at infinity in the indiscrete case. The Gelfand topology on $\hat{X}$ coincides with the compact open topology commonly used in the group theory. An explicit proof was not given but is not difficult.

The Banach algebra $\mathbf{B}$ has an involution defined by

$$f^*(x) = \overline{f(-x)}, \quad f \in L^1(X), \tag{1}$$

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