J. Math. Kyoto Univ. (JMKYAZ)
41-4 (2002), 627–667

Values of the Epstein zeta functions at the critical points

By

Akio FUJII

1. Introduction

We continue our study on the distribution of the zeros of the Epstein zeta functions attached to the positive definite quadratic forms. Here we are concerned only with the vanishing or the non-vanishing of the following type of Epstein zeta functions at the critical points. For each integer $K \geq 1$ and for any positive number d, we define $F_d(s, K)$ by

$$F_d(s, K) = \sum{}' \frac{1}{(m_1^2 + m_2^2 + \cdots + m_K^2 + d(m_{K+1}^2 + m_{K+2}^2 + \cdots + m_{2K}^2))^s},$$

for $\Re(s) > K$, the dash indicating that m_j's run over the integers excluding the case $(m_1, m_2, m_3, \ldots, m_{2K}) = (0, 0, 0, \ldots, 0)$. We have seen in Fujii [7] that $F_d(s, K)$ can be continued analytically to the whole complex plane with a simple pole at $s = K$ and has the following functional equation:

$$\left(\frac{\pi}{\sqrt{d}}\right)^{-s} \Gamma(s) F_d(s, K) = \left(\frac{\pi}{\sqrt{d}}\right)^{-(K-s)} \Gamma(K - s) F_d(K - s, K),$$

where $\Gamma(s)$ is the Γ-function. Thus the critical point of $F_d(s, K)$ is $s = K/2$. The purpose of the present article is to prove the following.

Main Theorem. *For all pairs (K, d) of integers $K \geq 1$ and positive real numbers d, we have*

$$F_d\left(\frac{K}{2}, K\right) \neq 0,$$

except exactly for eight pairs.

This is much sharper than what we have claimed and expected in our previous work Fujii [7].

Main Theorem consists of the following two theorems. Theorem 1 describes the exceptional eight cases. Theorem 2 describes that they are really exceptional.

Communicated by Prof. Y. Yoshida, June 14, 1999