J. Math. Kyoto Univ. (JMKYAZ)
42-1 (2002), 105–115

Spaces of polynomials with real roots of bounded multiplicity

By

Kohhei Yamaguchi

1. Introduction

The principal motivation for this paper derives from work of V. A. Vassiliev [16], [17], [18] and [19]. He describes a general method for calculating the cohomology of certain spaces of discriminants. For $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, we denote by $\mathrm{P}^d_n(\mathbb{K})$ the space consisting of all monic polynomials

$$f(z) = z^d + a_1 z^{d-1} + \cdots + a_{d-1} z + a_d \in \mathbb{K}[z]$$

of degree d which have no n-fold real roots (but may have complex ones of arbitrary multiplicity!). As his typical example, he takes the space $\mathrm{P}^d_n(\mathbb{R})$ and in particular, he computes the cohomology of the space $\mathrm{P}^d_n(\mathbb{R})$.

Theorem 1.1 (Vassiliev [16] and [17]). *If $n \geq 3$,*

$$H^j(\mathrm{P}^d_n(\mathbb{R}), \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if} \quad j = k(n-2), \quad 0 \leq k \leq [d/n], \\ 0 & \text{otherwise}, \end{cases}$$

where $[x]$ denotes the integer part of a real number x.

There is a "jet map" $j^d_n = j^d_{n;\mathbb{R}} : \mathrm{P}^d_n(\mathbb{R}) \to \Omega \mathbb{R}\,\mathrm{P}^{n-1}$ given by

$$j^d_n(f)(t) = \begin{cases} [f(t) : f'(t) : f''(t) : \cdots : f^{(n-1)}(t)] & \text{if} \quad t \in \mathbb{R}, \\ [1 : 0 : 0 : \cdots : 0] & \text{if} \quad t = \infty, \end{cases}$$

for $f \in \mathrm{P}^d_n(\mathbb{R})$ and $t \in S^1 = \mathbb{R} \cup \infty$.

If $n \geq 3$ and $k \in \mathbb{Z}/2 = \pi_0(\Omega \mathbb{R}\,\mathrm{P}^{n-1})$, we denote by $\Omega_k \mathbb{R}\,\mathrm{P}^{n-1}$ the space consisting of all base point preserving maps $f : S^1 \to \mathbb{R}\,\mathrm{P}^{n-1}$ with $[f] = k$. Remark that $j^d_n(\mathrm{P}^d_n(\mathbb{R})) \subset \Omega_{[d]_2} \mathbb{R}\,\mathrm{P}^{n-1}$, where $[d]_2$ denotes the number d mod 2. So it is regarded as the map $j^d_n : \mathrm{P}^d_n(\mathbb{R}) \to \Omega_{[d]_2} \mathbb{R}\,\mathrm{P}^{n-1} \simeq \Omega S^{n-1}$. For $\mathbb{K} = \mathbb{C}$, we can also define the jet map $j^d_{n;\mathbb{C}} : \mathrm{P}^d_n(\mathbb{C}) \to \Omega_{[d]_2}(\mathbb{C}^n - \{0\})/\mathbb{R}^* \simeq \Omega S^{2n-1}$ in a similar way.

Vassiliev also obtains the following result.

1991 *Mathematics Subject Classification(s).* 55P35, 58D15, 57R45
Communicated by Prof. A. Kono, June 27, 2000
Revised August 25, 2000