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A note on moduli of vector bundles on rational surfaces

By

Kōta Yoshioka

1. Introduction

Let (X, H) be a pair of a smooth rational surface X and an ample divisor H on X. Assume that $(K_X, H) < 0$. Let $\overline{M}_H(r, c_1, \chi)$ be the moduli space of semi-stable sheaves E such that $\mathrm{rk}(E) = r$, $c_1(E) = c_1$ and $\chi(E) = \chi$. The relationship between moduli spaces of different invariants is an interesting subject to be studied. If $(c_1, H) = 0$ and $\chi \leq 0$, then Maruyama [Ma2], [Ma3] studied such relations and constructed a contraction map $\phi : \overline{M}_H(r, c_1, \chi) \to \overline{M}_H(r-\chi, c_1, 0)$. Moreover he showed that the image is the Uhlenbeck compactification of the moduli space of μ-stable vector bundles. In particular, he gave an algebraic structure on the Uhlenbeck compactification which was topologically constructed before. After Maruyama's result, Li [Li] constructed the birational contraction for general cases, by using a canonical determinant line bundle, and gave an algebraic structure on the Uhlenbeck compactification. Although Maruyama's method works only for special cases, his construction is interesting of its own. Let us briefly recall his construction. Let E be a semi-stable sheaf such that $\mathrm{rk}(E) = r$, $c_1(E) = c_1$ and $\chi(E) = \chi$. Then $H^i(X, E) = 0$ for $i = 0, 2$. We consider the universal extension

$$0 \to E \to F \to H^1(X, E) \otimes \mathcal{O}_X \to 0. \tag{1.1}$$

Maruyama showed that F is a semi-stable sheaf such that $\mathrm{rk}(F) = r - \chi$, $c_1(F) = c_1$ and $\chi(F) = 0$. Then we have a map

$$\phi : \overline{M}_H(r, c_1, \chi) \to \overline{M}_H(r - \chi, c_1, 0). \tag{1.2}$$

He showed that ϕ is an immersion on the open subscheme consisting of μ-stable vector bundles and the image of ϕ is the Uhlenbeck compactification. For the proof, the rigidity of $\mathcal{O}_X$ is essential. In this note, we replace $\mathcal{O}_X$ by other rigid and stable vector bundles E_0 and show that similar results hold, if the E_0-twisted degree $\deg_{E_0}(E) := (c_1(E_0^\vee \otimes E), H) = 0$. If H is a general polarization, then we also show that $\mathrm{im}\,\phi$ is normal (Theorem 4.5).

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