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Dihedral covers and an elementary arithmetic on elliptic surfaces

By

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1. Introduction

Let X and Y be normal projective varieties defined over $\mathbb{C}$, the complex number field. We call X a cover of Y if there exists a finite surjective morphism $\pi : X \to Y$. The rational function field, $\mathbb{C}(X)$, is regarded as an algebraic extension of that of Y, $\mathbb{C}(Y)$, with $\deg \pi = [\mathbb{C}(X) : \mathbb{C}(Y)]$. The branch locus of a cover $\pi : X \to Y$, denoted by $\Delta(X/Y)$ or Δ_π, is the subset of Y given by

$$\Delta_\pi = \{y \in Y \mid \pi \text{ is not locally isomorphic over } y\}.$$

It is well-known that Δ_π is an algebraic subset of codimension 1 if Y is smooth ([19]). We call X a D_{2n}-cover if (*i*) $\mathbb{C}(X)/\mathbb{C}(Y)$ is Galois and (*ii*) $\mathrm{Gal}(\mathbb{C}(X)/\mathbb{C}(Y)) \cong D_{2n}$, the dihedral group of order $2n$. To present D_{2n}, we use the notation

$$D_{2n} = \langle \sigma, \tau \mid \sigma^2 = \tau^n = (\sigma\tau)^2 = 1\rangle,$$

and fix it throughout this article. Given a D_{2n}-cover $\pi : X \to Y$, we canonically obtain the double cover, $D(X/Y)$, of Y by taking the $\mathbb{C}(X)^\tau$-normalization of Y, where $\mathbb{C}(X)^\tau$ is the fixed field of $\langle\tau\rangle$. X is an n-cyclic cover of $D(X/Y)$ by its definition. We denote these covering morphisms by $\beta_1(\pi) : D(X/Y) \to Y$ and $\beta_2(\pi) : X \to D(X/Y)$, respectively. In [13], the author gave a method to deal with D_{2n}-covers. He exploited it in order to study D_{2n}-covers of $\mathbb{P}^2$ ([14], [15], and [16]) in the following setting:

(i) Y is a surface obtained by a succession of blowing-ups from $\mathbb{P}^2$.

(ii) $D(X/Y)$ has an elliptic fibration $\varphi : D(X/Y) \to \mathbb{P}^1$ with section O and $\beta_1(\pi) : D(X/Y) \to Y$ coincides with the quotient map induced by the inversion homomorphism $z \mapsto -z$ with respect to the group law.

(iii) X also has an elliptic fibration and $\beta_2(\pi)$ is the quotient map by the translation-by-n-torsion element in the Mordell-Weil group.

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