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On the Hecke operator U(p)

By

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(with an appendix by Ralf Schmidt)

Introduction

The operator $U(p)$ is a familiar tool in the theory of elliptic modular forms (introduced by Hecke in [5]). It can be defined in the same way on holomorphic Siegel modular forms of degree n for the congruence subgroups $\Gamma_0(N) := \left\{ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathrm{Sp}(n, \mathbb{Z}) \mid c \equiv 0\ (N) \right\}$; the action of $U(p)$ on the Fourier expansion of such a modular form f is given by

$$f = \sum_T a(T) e^{2\pi i \mathrm{tr}(TZ)} \longmapsto f \mid U(p) = \sum_T a(pT) e^{2\pi i \mathrm{tr}(TZ)};$$

here T runs over all symmetric half integral positive semidefinite matrices of size n and Z is an element of Siegel's upper half space. To be more precise, let us denote by $[\Gamma_0(N), k, \chi]$ the space of Siegel modular forms of weight k with respect to the group $\Gamma_0(N)$ and the nebentypus character χ. Then $U(p)$ maps this space into itself (if $p \mid N$) and maps it into $[\Gamma_0(\frac{N}{p}), k, \chi]$ if $p^2 \mid N$ and χ is defined modulo $\frac{N}{p}$. It is clear from the theory of old- and newforms (for $n = 1$) that we can expect a nontrivial kernel for $U(p)$ if $p^2 \mid N$ and χ is defined modulo $\frac{N}{p}$.

The injectivity of $U(p)$ for $p^2 \mid N$ and χ not defined modulo $\frac{N}{p}$ can be proved along the classical lines (see Section 6). The main purpose of the present note is to show that $U(p)$ is injective for $p \mid\mid N$ (see Section 3). This will be done in a purely algebraic manner by studying the properties (invertibility) of the double coset $\Gamma_0(p) \cdot \begin{bmatrix} 0_n & -\mathbf{1}_n \\ \mathbf{1}_n & 0_n \end{bmatrix} \cdot \Gamma_0(p)$ in the abstract Hecke algebra associated to the pair $(\Gamma_0(p), \mathrm{Sp}(n, \mathbb{Z}))$ and its analogue over the finite field $\mathbb{F}_p$; to include the case of nontrivial nebentypus χ we will have to work with a slightly smaller group $\Gamma_1(p)$.

The results of this paper are motivated (and are used in a crucial way) in our investigation of the basis problem for Siegel modular forms with level [2].

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