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Real K-homology of complex projective spaces

By

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Introduction

The real K-homology theory is one of a few examples of generalized homology theories which take values in the category of comodules over the associated Hopf algebroid, which are not complex oriented in the sense of Adams [1], namely the real K-cohomology of the infinite dimensional complex projective space does not have a structure of formal group law induced by the group structure $m : \boldsymbol{C}P^\infty \times \boldsymbol{C}P^\infty \to \boldsymbol{C}P^\infty$. However, the real K-homology of the infinite dimensional complex projective space has the Pontrjagin ring structure which is regarded as a virtual dual of non-existent structure of formal group law ([4]). From this point of view, the ring structure of the real K-homology of the infinite dimensional complex projective space might be of some interest. The aim of this paper is to determine the module structure of the real K-homology of complex projective spaces over the coefficient ring KO_* and to describe the ring structure of the real K-homology of the infinite complex projective space.

In the first section, we prepare some necessary results in the following sections. Next, we determine the "conjugation map" on $K_*(\boldsymbol{C}P^l)$ induced by the map $BU(n) \to BU(n)$ which classifies the complex conjugate of the canonical bundle. We make some analysis on the conjugation map in section three and define certain elements of $K_*(\boldsymbol{C}P^\infty)$ which generates the image of the complexification map $KO_*(\boldsymbol{C}P^\infty) \to K_*(\boldsymbol{C}P^\infty)$. In section four, we determine the KO_*-module structure of $K_*(\boldsymbol{C}P^l)$ by using the Atiyah-Hirzebruch spectral sequence. It turns out that the complexification map $\boldsymbol{c} : \widetilde{KO}_*(\boldsymbol{C}P^l) \to \widetilde{K}_*(\boldsymbol{C}P^l)$ is injective if l is even or ∞. By virtue of this fact, we can describe the ring structure of $KO_*(\boldsymbol{C}P^\infty)$ by examining the image of $\boldsymbol{c}$ in the last section.

1. Preliminaries

We first recall the Bott periodicity

$$O \simeq \Omega(\boldsymbol{Z} \times BO), \quad O/U \simeq \Omega O, \quad U/Sp \simeq \Omega(O/U), \quad \boldsymbol{Z} \times BSp \simeq \Omega(U/Sp)$$
$$Sp \simeq \Omega(\boldsymbol{Z} \times BSp), \quad Sp/U \simeq \Omega Sp, \quad U/O \simeq \Omega(Sp/U), \quad \boldsymbol{Z} \times BO \simeq \Omega(U/O).$$

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