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Positive continuous additive functionals of multidimensional Brownian motion and the Brownian local time

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1. Introduction

The local time of multidimensional Brownian motion was first introduced by Imkeller and Weisz [7]. They showed the existence of the limit $L(t,x)$ of $\int_0^t p_N(\varepsilon, W_s - x)ds$ as a generalized Wiener functional unless $x = 0$, where W_s denotes the N-dimensional Brownian motion starting from the origin and $p_N(s,y)$ the Gaussian kernel:

$$p_N(s,y) = \left(\frac{1}{\sqrt{2\pi s}}\right)^N \mathrm{e}^{-|y|^2/2s}, \qquad (s > 0, y \in \mathbb{R}^N).$$

Let $\varphi \in \mathcal{D}$, a smooth function on $\mathbb{R}^N$ with compact support, satisfy $\int \varphi(y)dy = 1$. Put $\varphi_\varepsilon(y) = \varphi(y/\varepsilon)/\varepsilon^N$. Then, using the same technique we also find that $\int_0^t \varphi_\varepsilon(W_s - x)ds$ has the same limit. We note that the functions $p_N(\varepsilon, y - x)$ and $\varphi_\varepsilon(y - x)$ approximate the delta function at x. We call the limit $L(t,x)$ the local time of N-dimensional Brownian motion.

The local time is interpreted as a generalized Wiener functional corresponding to the delta function. Now we are interested in the existence of a generalized Wiener functional corresponding to another positive distribution T. To explain more rigorously, we determine the limit point of $\int_0^t T * \varphi_\varepsilon(W_s + x)ds$ under some conditions. Here $T * \varphi(y) = \int \varphi(y - x)\mu_T(dx)$, $\mu_T(dx)$ denoting the corresponding measure of T, i.e., $\langle T, \varphi\rangle = \int \varphi(x)\mu_T(dx)$. We also discuss on the integral representation of this functional using this measure μ_T and the Brownian local time. Details are discussed in Section 3.

Next we consider positive continuous additive functionals (PCAF in abbreviation) of N-dimensional Brownian motion. In the case where $N = 1$, one of the most typical additive functional is the local time, and every PCAF of the Brownian motion can be represented by the integral of the local time with respect to the Revuz measure associated to the PCAF(see, for instance,

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