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On Samelson products in $\mathrm{Sp}(n)$

By

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1. Introduction

We work in the category of based spaces and based maps. For based spaces X, Y, $[X, Y]$ denotes the based homotopy set of maps from X to Y. When Y is a group-like space, we will always assume $[X, Y]$ is a group by the natural group structure. We denote Samelson products by $\langle -, - \rangle$. Let Q_n be the symplectic quasi-projective space of rank n, that is, the image of the map $\psi_n \colon S^{4n-1} \times S^3 \to \mathrm{Sp}(n)$ defined by the condition

$$\psi_n(x, \lambda)(v) = \begin{cases} v, & v \text{ is perpendicular to } x \\ x\lambda, & v = x \end{cases}$$

for $v \in \mathbf{H}^n$, where we identify S^{4k-1} with the unit sphere in $\mathbf{H}^k$ and $\mathbf{H}^n$ has the right scalar multiplication. Then Q_n has the cell structure $S^3 \cup e^7 \cup e^{11} \cup \cdots \cup e^{4n-1}$. We denote the inclusion $Q_n \to \mathrm{Sp}(n)$ by ϵ_n. Hamanaka, Kaji and Kono [4] determined the order of $\langle \epsilon_2, \epsilon_2 \rangle$ at the prime three. Hamanaka [3] also studies the Samelson products in the unitary groups localized at a prime p. In this note, we first determine the order of $\langle \epsilon_2, \epsilon_2 \rangle$, not at the prime three, by modifying the calculation in [4].

Theorem 1.1. *The order of the Samelson product* $\langle \epsilon_2, \epsilon_2 \rangle$ *is* 280.

This implies the following result:

Corollary 1.1 ([5]). $\mathrm{Sp}(2)_{(3)}$ *is homotopy commutative, where* $-_{(p)}$ *denotes the localization at a prime p in the sense of* [2].

Generalizing this calculation, we discuss the order of $\langle \epsilon_n, \epsilon_n \rangle$ at an odd prime in a certain range of n which is given by p. Using this, we determine the order of $\langle \epsilon_n, \epsilon_n \rangle$ for $n = 3, 4, 5$ at all primes but 2.

Theorem 1.2. *At odd primes, the order of the Samelson products* $\langle \epsilon_3, \epsilon_3 \rangle$, $\langle \epsilon_4, \epsilon_4 \rangle$ *and* $\langle \epsilon_5, \epsilon_5 \rangle$ *are* $31185 = 3^4 \cdot 5 \cdot 7 \cdot 11$, $6081075 = 3^5 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$ *and* $68746552875 = 3^5 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19$ *respectively.*

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