Illinois Journal of Mathematics
Volume 54, Number 1, Spring 2010, Pages 11–18
S 0019-2082

A Strengthened Freiheitssatz

Paul E. Schupp*
Department of Mathematics, University of Illinois, Urbana, Ill. 61801, USA

The basic result in the theory of one relator groups is, of course, the Freiheitssatz of Magnus [5]. In the case where the defining relator is a proper power, the "Spelling Theorem" of Newman [8, 9] gives sharper results. At the 1974 Calgary Conference on infinite group theory, Steve Pride told me that Gurevich [1] had strengthened Newman's theorem. Steve asked whether this result could be further improved. Reflection on the matter led to the discovery that there is a single theorem which strengthens both the Freiheitssatz for one relator groups in general and Newman's results in the torsion case.

We state the general theorem below. Perhaps the most interesting consequence is the following. Let $G=\langle a, b, c, \ldots; r\rangle$ where r is cyclically reduced. Let R^* be the symmetrized set generated by r, that is, R^* consists of all cyclic permutations of $r^{\pm 1}$. If u is a non-trivial freely reduced word such that $u=1$ in G, then u has a subword s which contains all the generators occurring in r, and such that s is also a subword of an element of R^*.

We state the general theorem in an "equational form".

Theorem. *Let $G=\langle a, b, c, \ldots; r\rangle$ where r is cyclically reduced. Write $r=z^n$, $n\geqq 1$, where z is not a proper power in the free group on $a, b, c, \ldots$. (Elements of the symmetrized set R^* generated by r thus have the form $(z^*)^n$ where z^* is a cyclic permutation of $z^{\pm 1}$.) If an equation $u=v$ holds in G where u and v are freely reduced words and v omits a generator which occurs in both r and u, then u contains a subword t of an element of R^* such that $t\equiv(z^*)^{n-1}s$ and s contains every generator which occurs in r but not v. (If $n=1$, then t is simply s.)*

The case $n>1$ is announced in Gurevich [1].

The amount of additional information which the theorem yields depends, of course, on the form of the defining relator r. It is interesting to note that a "small cancellation" type of conclusion follows for some groups purely by one-relator methods. For example, if $G=\langle a_1, \ldots, a_g; a_1^2 \ldots a_g^2=1\rangle$, then a subword containing all the generators must be of length at least $2(g-1)$.

* This research was supported by the National Science Foundation and a sabbatical leave from the University of Illinois.