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ON SANOV 4TH-COMPOUNDS OF A GROUP

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Dedicated to the memory of my teacher Professor Reinhold Baer

1. Introduction

In his elegant inductive proof that every finitely generated group of exponent 4 is finite, Sanov used the following construction.

Let M be a group and let u be an involution in M. We form a group $S_u(M,a)$ by means of the relations $a^2 = u$ and $(ma)^4 = 1$ for every $m \in M$. When $u = 1$, we write $S_0(M,a)$ for the corresponding group.

We call $S_u(M,a)$ a Sanov compound and there is one for every conjugacy class of involutions in M. Sanov proved that for finite M of order m, every Sanov compound $S_u(M,a)$ has finite order at most m^{m+1}. (See, for example, [2, Theorem 18.3.1] or [3, Theorem 14.2.4].) Here we establish some general results concerning $S_u(M,a)$. For example, if M is infinite cyclic, then $S_0(M,a)$ is the extension of a countable elementary abelian 2-group by the infinite dihedral group. If M is cylic of order 3, then $S_0(M,a)$ is isomorphic to S_4. For $M = A_4$, $S_0(M,a)$ has order $2^9 \cdot 3$, while $S_u(M,a)$ has order $2^6 \cdot 3$ for $u = (1,2)(3,4)$.

For computational purposes one uses a presentation for M via generators and relations. Then one adds the extra relations defining $S_u(M,a)$. These extra relations usually induce further relations in M. Thus, while M itself may not be a subgroup of $S_u(M,a)$, there exists a normal subgroup K_u of M such that $S_u(M,a)$ is isomorphic to $S_{\overline{u}}(\overline{M},a)$, where $\overline{M} = M/K_u$ belongs to $S_{\overline{u}}(\overline{M},a)$. For example, when M is a dihedral group of order $2n$, with n odd, $S_0(M,a) = S_0(C_2,a)$ is dihedral of order 8 and $K_0 = M'$, the commutator subgroup of M. We also show that for M finite, simple and non-abelian, $S_u(M,a) = S_0(1,a)$ is cyclic of order 2. Originally these investigations were prompted by a remark of M. Newman who asked if every Sanov compound of a 2-group M is itself a 2-group. We give a positive answer to this question, and a bound for the order. In a later paper we will examine the compounds of soluble groups and present further information on the groups M/K_u.

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