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KILLING FIELDS, MEAN CURVATURE, TRANSLATION MAPS

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Abstract. D. Hoffman, R. Osserman and R. Schoen proved that if the Gauss map of a complete constant mean curvature (cmc) oriented surface M immersed in $\mathbb{R}^3$ is contained in a closed hemisphere of $\mathbb{S}^2$ (equivalently, the function $\langle \eta, v\rangle$ does not change sign on M, where η is a unit normal vector of M and v some non-zero vector of $\mathbb{R}^3$), then M is invariant by a one parameter subgroup of translations of $\mathbb{R}^3$ (the one determined by v). We obtain an extension of this result to the case that the ambient space is a Riemannian manifold N and M is a hypersurface on N by requiring that the function $\langle \eta, V\rangle$ does not change sign on M, where V is a Killing field on N. We also obtain a stability criterium for cmc surfaces in N^3. In the last part of the article we consider a Killing parallelizable Riemannian manifold N and define a translation map $\gamma : M \to \mathbb{R}^n$ of a hypersurface M of N which is a natural extension of the Gauss map of a hypersurface in $\mathbb{R}^n$. Considering the same hypothesis on the image of γ we obtain an extension to this setting of the original Hoffmann-Osserman-Schoen result. Motivated by this extension, we restate in this context a conjecture made by M. P. do Carmo which, in $\mathbb{R}^3$, states that the Gauss image of a complete cmc surface which is not a plane nor a cylinder contains a neighborhood of some equator of the sphere.

1. Introduction

D. Hoffman, R. Osserman and R. Schoen proved that if the Gauss map of a complete constant mean curvature (cmc) oriented surface M immersed in $\mathbb{R}^3$ is contained in a closed hemisphere of $\mathbb{S}^2$, then M is invariant by a one parameter subgroup of translations of $\mathbb{R}^3$; it then follows that M is a circular cylinder or a plane (Theorem 1 of [HOS]). This result may be equivalently stated as follows: Let η be a unit normal vector field to M in $\mathbb{R}^3$. If, for some nonzero vector $V \in \mathbb{R}^3$, the map

$$f(p) := \langle \eta(p), V\rangle\,,\ p \in M, \tag{1.1}$$

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