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VECTOR-VALUED MODULAR FORMS AND POINCARÉ SERIES

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Abstract. We initiate a general theory of vector-valued modular forms associated to a finite-dimensional representation ρ of $SL(2, \mathbf{Z})$. We introduce vector-valued Poincaré series and Eisenstein series and a version of the Petersson inner product, and establish analogs of basic results from the classical theory of modular forms concerning these objects, at least if the weight is large enough. In particular, we show that the space of entire vector-valued modular forms of weight k associated to ρ is a finite-dimensional vector space which, for large enough k, is nonzero and spanned by Poincaré series. We show that Hecke's estimate $a_n = O(n^{k-1})$ continues to apply to the Fourier coefficients of component functions of entire vector-valued modular forms associated to ρ for large enough k.

1. Introduction

Vector-valued modular forms have been a feature of the theory of modular forms for some time, although there seems to be no systematic development of the subject in the literature. Selberg already pointed out in [S] how they could be used in estimating the growth of Fourier coefficients of scalar modular forms. Vector-valued forms arise naturally also in the work of Eichler and Zagier on Jacobi forms [EZ, Chapter II]. In a previous paper [KM1] we considered in detail a particular class of vector-valued modular forms, and in [KM2] we obtained Hecke-type estimates on the Fourier coefficients of a vector-valued modular form associated to a representation $\rho : \Gamma \to GL(p, \mathbf{C})$. Building on [KM2], the present paper develops some of the foundations of the theory of vector-valued modular forms.

Suppose we are given a multiplier system v in weight k on Γ. As we will explain in Section 2, it is both natural and convenient to restrict consideration

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