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DUALIZING COMPLEX OF THE INCIDENCE ALGEBRA OF A FINITE REGULAR CELL COMPLEX

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ABSTRACT. Let Σ be a finite regular cell complex with $\emptyset \in \Sigma$, and regard it as a *poset* (i.e., partially ordered set) by inclusion. Let R be the incidence algebra of the poset Σ over a field k. Corresponding to the Verdier duality for constructible sheaves on Σ, we have a dualizing complex $\omega^\bullet \in D^b(\mathrm{mod}_{R\otimes_k R})$ giving a duality functor from $D^b(\mathrm{mod}_R)$ to itself. This duality is somewhat analogous to the Serre duality for a projective scheme ($\emptyset \in \Sigma$ plays a role similar to that of "irrelevant ideals"). If $H^i(\omega^\bullet) \neq 0$ for exactly one i, then the underlying topological space of Σ is Cohen-Macaulay (in the sense of the Stanley-Reisner ring theory). The converse also holds if Σ is a simplicial complex. R is always a Koszul ring with $R^! \cong R^{\mathsf{op}}$. The relation between the Koszul duality for R and the Verdier duality is discussed.

1. Introduction

Let Σ be a finite regular cell complex, and $X := \bigcup_{\sigma\in\Sigma} \sigma$ its underlying topological space. The order given by $\sigma > \tau \overset{\text{def}}{\Longleftrightarrow} \bar{\sigma} \supset \tau$ makes Σ a finite partially ordered set (*poset*, for short). Here $\bar{\sigma}$ is the closure of σ in X. Let R be the incidence algebra of the poset Σ over a field k. For a ring A, mod_A denotes the category of finitely generated left A-modules. In this paper, we study the bounded derived category $D^b(\mathrm{mod}_R)$ using the theory of constructible sheaves (e.g., Poincaré-Verdier duality). For the sheaf theory, consult [6], [7], [14]. We basically use the same notation as [6].

Let $\mathrm{Sh}_c(X)$ be the category of k-constructible sheaves on X with respect to the cell decomposition Σ. We have an exact functor $(-)^\dagger : \mathrm{mod}_R \to \mathrm{Sh}_c(X)$. For $M \in \mathrm{mod}_R$, we have a natural decomposition $M = \bigoplus_{\sigma\in\Sigma} M_\sigma$ as a k-vector space. If $p \in \sigma \subset X$, the stalk $(M^\dagger)_p$ of $M^\dagger$ at the point p is isomorphic to M_σ.

Let $\Sigma' := \Sigma \setminus \emptyset$ be an induced subposet of Σ, and T the incidence algebra of Σ' over k. Then we have a category equivalence $\mathrm{mod}_T \cong \mathrm{Sh}_c(X)$, which is well

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