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CONTRACTIVE AND COMPLETELY CONTRACTIVE HOMOMORPHISMS OF PLANAR ALGEBRAS

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ABSTRACT. We consider contractive homomorphisms of a planar algebra $\mathcal{A}(\Omega)$ over a finitely connected bounded domain $\Omega \subseteq \mathbb{C}$ and ask if they are necessarily completely contractive. We show that a homomorphism $\rho : \mathcal{A}(\Omega) \to \mathcal{B}(\mathcal{H})$ for which $\dim(\mathcal{A}(\Omega)/\ker \rho) = 2$ is the direct integral of homomorphisms ρ_T induced by operators on two-dimensional Hilbert spaces via a suitable functional calculus $\rho_T : f \mapsto f(T)$, $f \in \mathcal{A}(\Omega)$. It is well known that contractive homomorphisms ρ_T induced by a linear transformation $T : \mathbb{C}^2 \to \mathbb{C}^2$ are necessarily completely contractive. Consequently, using Arveson's dilation theorem for completely contractive homomorphisms, one concludes that such a homomorphism ρ_T possesses a dilation. In this paper, we construct this dilation explicitly. In view of recent examples discovered by Dritschel and McCullough, we know that not all contractive homomorphisms ρ_T are completely contractive even if T is a linear transformation on a finite-dimensional Hilbert space. We show that one may be able to produce an example of a contractive homomorphism ρ_T of $\mathcal{A}(\Omega)$ which is not completely contractive if an operator space which is naturally associated with the problem is not the MAX space. Finally, within a certain special class of contractive homomorphisms ρ_T of the planar algebra $\mathcal{A}(\Omega)$, we construct a dilation.

1. Introduction

All our Hilbert spaces are over complex numbers and are assumed to be separable. Let $T \in \mathcal{B}(\mathcal{H})$, the algebra of bounded operators on $\mathcal{H}$. The operator T induces a homomorphism $\rho_T : p \mapsto p(T)$, where p is a polynomial. Equip the polynomial ring with the supremum norm on the unit disc, that is, $\|p\| = \sup\{|p(z)| : z \in \mathbb{D}\}$. A well-known inequality due to von Neumann (cf. [18]) asserts that ρ_T is contractive, that is, $\|\rho_T\| \leq 1$, if and only if the operator

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