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BOUNDEDNESS OF COMMUTATORS OF FRACTIONAL AND SINGULAR INTEGRALS FOR THE EXTREME VALUES OF p

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1. Introduction

It is well known that commutators of singular integrals with multiplication by a measurable function $b(x)$ are bounded operators on L^p, $1 < p < \infty$, as long as b is a *BMO* function [C-R-W].

Moreover if the commutator of all Riesz transforms are bounded for some p, $1 < p < \infty$, the function b must necessarily belong to *BMO*. Similar results are also known for the fractional integral operators I_α in connection with the boundedness from L^p into L^q, $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$ [Ch].

Later on, Segovia and Torrea [S-T] have considered this problem in the more general context of vector valued operators including in this approach, commutators associated for example to maximal functions.

It this paper we find sufficient conditions on the function b in order to obtain $H^1 \to L^1$ and $L^{n/\alpha} \to BMO$ boundedness of such commutators. In most of the cases the given conditions will be also necessary. See [P] for a discussion in the case b is a *BMO* function. We have chosen to work in the general context of vector valued operators of singular integral type as to include a larger class of commutators. Following this line, we first prove two general theorems (Theorems A and B in Section 2) expressing the conditions on b in terms of the kernel of the given operator. Afterwards, in Section 3, we apply our theorems to some particular cases like the Hilbert transform, fractional integrals and maximal operators of smooth approximations to the identity.

As an example, commutators with the Hilbert transform are bounded from H^1 into L^1 only in the trivial case that b equals a constant; this is also the case in the other extreme, L^∞ into *BMO*. Similar results are proven for the fractional integral, therefore since a constant function corresponds to the zero function in *BMO* we have that for non-zero *BMO* functions the commutator with the Hilbert transform or fractional integral is *not* bounded in the extreme cases; see Theorems (3.1) and (3.10). The picture improves in the periodic case. In fact we prove that commutators with the conjugate function are bounded from L^∞ into *BMO* if and only if b belongs to a class a little bit more restricted than *BMO*, the space BMO_φ for $\varphi(t) = |\log t|^{-1}$.

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