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# CHARACTERIZING HILBERT SPACE FRAMES WITH THE SUBFRAME PROPERTY[1]

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## 1. Introduction

A sequence $(f_i)_{i=1}^{\infty}$ in a Hilbert space $H$ which is a frame for its closed linear span is called a *frame sequence*. If every subsequence of $(f_i)_{i=1}^{\infty}$ is a frame sequence, we say that the frame has the *subframe property*. If $(f_i)_{i=1}^{\infty}$ is a frame for $H$ with the subframe property and additionally there are uniform upper and lower frame bounds for all subsequences of the frame, then we call $(f_i)_{i=1}^{\infty}$ a *Riesz frame*. Riesz frames were introduced in [6] where it was shown that every Riesz frame for $H$ contains a subset which is a Riesz basis for $H$. The projection methods [4] play a central role in evaluating truncation error which arises in computing approximate solutions to moment problems, as well as handling the very difficult problem of computing dual frames. There were many natural questions arising from the literature concerning the interrelationships between Riesz frames, frames with the subframe property, and the projection methods [2], [4], [5], [6], [8]. In this paper we characterize Riesz frames and frames with the subframe property which allows us to answer most of these questions.

## 2. Riesz frames

If $\mathcal{F}$ is a subset of $H$, we write *span* $\mathcal{F}$ for the closed linear span of $\mathcal{F}$. A sequence $(f_i)_{i=1}^{\infty}$ in $H$ is called a *frame* for $H$ if there are positive constants $A$, $B$ satisfying

$$A\|f\|^2 \leq \sum_{i=1}^{\infty} |\langle f, f_i\rangle|^2 \leq B\|f\|^2, \quad \forall f \in H. \tag{2.1}$$

We call $A$, $B$ the lower and upper frame bounds respectively. In general, a subset of a frame need not be a frame for its closed linear span. But clearly $B$ is an upper frame bound for every subset of the frame (i.e. It is only the lower frame bound that might be lost when switching to a subset of a frame). For a Riesz frame, the common frame bounds for all subsets of the frame will be called the *Riesz frame bounds*. The

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