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EXTREMAL PROPERTIES OF HILBERT FUNCTIONS

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1. Introduction

Recently there has been a lot of interest in the extremal properties of Hilbert functions. This subject is related to combinatorics, commutative algebra, and algebraic geometry. It was founded by Macaulay [12] who gave a characterization of the Hilbert functions of quotients of polynomial rings. His result can also be interpreted as a characterization of the h-vectors of multicomplexes [15, §2.2]. Kruskal [11] and Katona [10] characterized the f-vectors of simplicial complexes, or equivalently, the Hilbert functions of quotients of exterior algebras. Gotzmann proved a Persistence Theorem which states that an extremal (in the sense of Macaulay's theorem) vector space of homogeneous polynomials of degree d generates an extremal vector space in degree $d+1$ [6]. We will call such a vector space *Gotzmann*. Green [7] characterized the Hilbert functions of rings obtained by moding out quotients of polynomial rings with fixed Hilbert function by a general linear form. Recently, Aramova, Herzog, and Hibi [1] proved a Persistence Theorem for exterior algebras.

In §2 we introduce some notation. In §3 we study Gotzmann vector spaces and obtain:

- a Reverse Persistence Theorem similar to Gotzmann's;
- a Persistence Theorem for vector spaces which are extremal in the sense of Green's theorem;
- a structure theorem for Gotzmann vector spaces which generalizes structure results of Green [7] and Bigatti-Geramita-Migliore [4].

Macaulay's theorem can be stated in two equivalent ways: one is that for every homogeneous ideal there is a lexicographic ideal with the same Hilbert function; the other is numerical. The corresponding generalizations to modules over polynomial rings however are not equivalent. Hulett [8, 9] and Pardue [13, 14] showed that for every graded submodule of a free module over a polynomial ring there is a lexicographic submodule with the same Hilbert function.

In §4 we give a numerical generalization of Macaulay's theorem and generalizations of Green's and Gotzmann's theorems for finitely generated modules over polynomial rings. We also give generalizations of Kruskal-Katona's theorem and Aramova-Herzog-Hibi Persistence Theorem for finitely generated modules over exterior algebras.

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