HOMOLOGY OF FUNCTION SPECTRA

BY

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1. Introduction

The purpose of this paper is the investigation of the singular homology groups of $F(X, Y)$, the space of base-point preserving maps from X to Y endowed with the compact-open topology [7]. We restrict ourselves to the case where X is compact and Y has the homotopy type of a countable CW-complex.

This problem was investigated by Borsuk [2] who studied the first nonzero Betti number of $F(X, S^m)$. Later, Moore [16] calculated the reduced singular integral homology groups $\tilde{H}_n(F(X, S^m))$ in the stable range. (We suppress notation of the coefficient group in case of integer coefficients.) Moore's result as restated by Spanier [20] in the language of spectra [13] says that $\tilde{H}_{-n}(\mathbf{F}(X, \mathbf{S})) \approx \tilde{H}^n(X)$, where $\mathbf{S}$ is the spectrum of spheres.

The crucial part of Moore's proof is that the homology of $F(X, S^m)$ defines a cohomology theory on X in the stable range. It is shown here that if $\mathbf{E}$ is a spectrum, the groups $\bar{H}^n(X) = \tilde{H}_{-n}(\mathbf{F}(X, \mathbf{E}))$ define a generalized cohomology theory [24] for finite complexes X. From a theorem of Brown [3], it follows that there is a spectrum $\mathbf{F}$ such that $\bar{H}^n(X) \approx \tilde{H}^n(X; \mathbf{F})$, the nth cohomology group of X with coefficients in the spectrum $\mathbf{F}$. This implies that the homotopy groups of $\mathbf{F}$ are isomorphic to the homology groups of $\mathbf{E}$. This suggests that $\mathbf{F}$ might be the infinite symmetric product of $\mathbf{E}$ and this is indeed the case. A final calculation arrives at the formula (Theorem (7.8)):

$$\tilde{H}_n(\mathbf{F}(X, \mathbf{E})) \approx \textstyle\sum_r \tilde{H}^{r-n}(X; \tilde{H}_r(\mathbf{E})).$$

It is assumed that the reader is familiar with the results and notation of Sections 1 through 5 of [24] which present the basic notions of spectra and generalized cohomology theories.

Sections 2 and 3 present elementary results on spectra and generalized cohomology theories. In Section 4 it is proved that the groups $\bar{H}^n(X)$ define a generalized cohomology theory for finite complexes. Section 5 introduces the notion of the infinite symmetric product $SP^\infty\mathbf{E}$ of a spectrum $\mathbf{E}$ and in Section 6 it is proved that $\bar{H}^n(X) \approx \tilde{H}^n(X; SP^\infty\mathbf{E})$. Sections 7 and 8 are devoted to the calculation of $\tilde{H}^n(X; SP^\infty\mathbf{E})$. In the final section, the results are applied to function spaces (rather than function spectra) and to the case where X is an arbitrary compact space.

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