POWER SERIES WHOSE SECTIONS HAVE ZEROS OF LARGE MODULUS. II

BY

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1. Introduction

Given a power series $\sum_{p=0}^{\infty} a_p z^p$, for each positive integer n let $\mathfrak{r}_n$ denote the smallest modulus of a zero of $\sum_{p=0}^{n} a_p z^p$, the n^{th} partial sum. Various growth properties of the sequence $\{\mathfrak{r}_n\}$ were discussed in [2]; since the present paper is an extension of several of these results, some familiarity with [2] is desirable.

If $\sum_{p=0}^{\infty} a_p z^p$ has a zero in the interior of its circle of convergence, then Hurwitz' theorem guarantees that $\{\mathfrak{r}_n\}$ converges to the smallest modulus of such a zero. If we note that $\mathfrak{r}_n \leq |a_0/a_n|^{1/n}$, then Hurwitz' theorem can be used to show that $\mathfrak{r}_n \to \infty$ if and only if $\sum_{p=0}^{\infty} a_p z^p = \exp\{g(z)\}$ for an entire function $g(z)$. There is in this case an interesting connection between the growth of $\{\mathfrak{r}_n\}$ and that of the maximum modulus of $g(z)$. In [2] it was shown that the condition

$$\limsup_{n\to\infty} \frac{\log n}{\mathfrak{r}_n^c} \leq d, \qquad 0 < c < \infty, 0 \leq d < \infty, \tag{1.1}$$

is satisfied if and only if

$$\sum_{p=0}^{\infty} a_p z^p = \exp\{g(z)\}, \qquad g(z) \textit{ an entire function of growth } (c,\, d). \tag{1.2}$$

(The statement that an entire function $g(z)$ is of growth (c, d) means that the order of $g(z)$ does not exceed c, and that the type of $g(z)$ does not exceed d if $g(z)$ is of order c.)

For each $d_1 > d$, (1.1) requires that $\{\mathfrak{r}_n\}$ should grow at least as rapidly as

$$\left[\frac{\log n}{d_1}\right]^{1/c}.$$

We shall investigate the possibility of replacing (1.1) by a weaker condition in which only a certain subsequence of $\{\mathfrak{r}_n\}$ is required to grow this rapidly.

One theorem of this type was obtained in [2]. There it was shown that if $c > 0$ and $\mathfrak{r}_n > n^c$ for infinitely many n, then $\sum_{p=0}^{\infty} a_p z^p = \exp\{P(z)\}$ for some polynomial $P(z)$ of degree $1/c$ or less. *No corresponding result is obtainable if n^c is replaced by a function of slower growth.* Specifically, if $\varphi(n)$ is a positive function such that $\varphi(n) = n^{o(1)}$ as $n \to \infty$, one can construct a power series $\sum_{p=0}^{\infty} a_p z^p$ of arbitrary convergence radius such that $\mathfrak{r}_n > \varphi(n)$ for infinitely many n. Such a construction is carried out in §3.

Results similar to (1.2) are obtainable if it is assumed that the values of n

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