ON OPTIMAL STOPPING RULES FOR s_n/n

BY

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1. Introduction

Let

$$x_1, x_2, \cdots \tag{1}$$

be a sequence of independent, identically distributed random variables on a probability space $(\Omega, \mathfrak{F}, P)$ with

$$P(x_1 = 1) = P(x_1 = -1) = \tfrac{1}{2}, \tag{2}$$

and let $s_n = x_1 + \cdots + x_n$. Let $i = 0, \pm 1, \cdots$ and $j = 0, 1, \cdots$ be two fixed integers. Assume that we observe the sequence (1) term by term and can decide to stop at any point; if we stop with x_n we receive the reward $(i + s_n)/(j + n)$. What stopping rule will maximize our expected reward?

Formally, a stopping rule t of (1) is any positive integer-valued random variable such that the event $\{t = n\}$ is in $\mathfrak{F}_n$ $(n \geq 1)$ where $\mathfrak{F}_n$ is the Borel field generated by $x_1, \cdots, x_n$. Let T denote the class of all stopping rules; for any t in T, s_t is a well-defined random variable, and we set

$$v_j(i \mid t) = E\left(\frac{i + s_t}{j + t}\right), \qquad v_j(i) = \sup_{t \in T} v_j(i \mid t). \tag{3}$$

It is by no means obvious that for given i and j there exists a stopping rule $\mathfrak{Z}_j(i)$ in T such that

$$v_j(i \mid \mathfrak{Z}_j(i)) = v_j(i) = \max_{t \in T} v_j(i \mid t); \tag{4}$$

such a stopping rule of (1) will be called *optimal* for the reward sequence

$$\frac{i + s_1}{j + 1}, \quad \frac{i + s_2}{j + 2}, \cdots. \tag{5}$$

Theorem 1 below asserts that for every $i = 0, \pm 1, \cdots$ and $j = 0, 1, \cdots$ there exists an optimal stopping rule $\mathfrak{Z}_j(i)$ for the reward sequence (5).

We remark that for any t in T and any $i = 0, \pm 1, \cdots$ and $j = 0, 1, \cdots$ the random variable

$$\begin{aligned} t' &= t && \text{if} \quad i + s_t \geq 1, \\ &= \text{first } n > t \text{ such that } i + s_n = 1 && \text{if} \quad i + s_t \leq 0 \end{aligned} \tag{6}$$

is in T and

$$i + s_{t'} \geq 1, \qquad 0 < E\left(\frac{i + s_{t'}}{j + t'}\right) \geq E\left(\frac{i + s_t}{j + t}\right). \tag{7}$$

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