# HOMOMORPHISMS OF IDEALS IN GROUP ALGEBRAS[1]

BY
FRANK FORELLI

**1.** Let $G_1$ and $G_2$ be locally compact abelian groups, let $J$ be a closed ideal in the group algebra $L^1(G_1)$, and let $T$ be a homomorphism of the ideal $J$ into the algebra $M(G_2)$ of bounded and regular Borel measures on $G_2$. The purpose of this paper is to show that when $\| T \| \leq 1$, then $T$ must come from an affine transformation of the dual group $\Gamma_2$ of $G_2$ into the dual group $\Gamma_1$ of $G_1$. When $J = L^1(G_1)$, this is known and is due to Helson [3]. Helson assumes also that $T$ is an isomorphism onto $L^1(G_2)$, but with a certain modification his argument works without this additional assumption. That $T$ comes from an affine transformation when $J = L^1(G_1)$ and $\| T \| \leq 1$ is also a corollary of the deep result of Cohen [1], [4, ch. 4] that every homomorphism of $L^1(G_1)$ into $M(G_2)$ comes from a piecewise affine transformation of $\Gamma_2$ into $\Gamma_1$. Although our extension of Helson's theorem is very modest and the proof we offer is not difficult, it does not seem to be possible to obtain this extension from either the results or the arguments of Helson and Cohen.

Let $\Delta_1$ be the open set of $\chi$ in $\Gamma_1$ such that $\hat{f}(\chi) \neq 0$ for some $f$ in $J$, where $\hat{f}$ is the Fourier transform of $f$. Then $\Delta_1$ can be identified with the maximal ideal space of $J$. Each $\chi$ in $\Delta_1$ defines a nontrivial complex homomorphism of $J$ whose value at $f$ in $J$ is $\hat{f}(\chi)$, and every such homomorphism of $J$ is obtained in this way. Moreover, $\hat{J}$ separates points on $\Delta_1$ as $J$ contains every $f$ in $L^1(G_1)$ such that the support of $\hat{f}$ is contained in $\Delta_1$ [4, p. 161], and the topology of $\Delta_1$ as a subspace of $\Gamma_1$ is the same as the topology induced on $\Delta_1$ by the functions in $\hat{J}$. Let $\Delta_2$ be the open set of $\chi$ in $\Gamma_2$ such that $(Tf)^\wedge(\chi) \neq 0$ for some $f$ in $J$, where $(Tf)^\wedge$ is the Fourier-Stieltjes transform of the measure $Tf$. Each $\chi$ in $\Delta_2$ defines a nontrivial complex homomorphism of $J$ whose value at $f$ in $J$ is $(Tf)^\wedge(\chi)$, and therefore there is $\varphi(\chi)$ in $\Delta_1$ such that $(Tf)^\wedge(\chi) = \hat{f}(\varphi(\chi))$. The map $\varphi$ from $\Delta_2$ into $\Delta_1$ defined in this way is continuous and we have for $f$ in $J$,

$$(Tf)^\wedge = 0 \qquad \text{on } \Gamma_2 \backslash \Delta_2$$

$$(Tf)^\wedge = \hat{f}(\varphi) \qquad \text{on } \Delta_2 .$$

Let $\Sigma_2$ be the coset in $\Gamma_2$ generated by $\Delta_2$. A map $\pi$ from $\Sigma_2$ into $\Gamma_1$ is said to be affine if

$$\pi(\alpha\beta\gamma^{-1}) = \pi(\alpha)\pi(\beta)\pi(\gamma)^{-1}$$

for all $\alpha$, $\beta$, $\gamma$ in $\Sigma_2$. Because the norm of a multiplicative linear functional on $J$ is 1, we always have $\| T \| \geq 1$ (unless $T = 0$, and we will always assume $T \neq 0$). We will show:

Received January 18, 1964.

[1] This work was supported by the National Science Foundation.