NILPOTENCY CONDITIONS FOR FINITE LOOPS

BY

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Introduction

The purpose of this note is to consider various properties of loops which are related to central nilpotency, and to determine some of the implications which hold among them. Since some of the properties considered are equivalent to nilpotency for the class of finite groups, it is natural to ask whether or not any of the properties is equivalent to central nilpotency for an interesting class of finite loops. Another reason for studying the problem is that the standard, group-theoretic proofs in the area of nilpotency ultimately depend on the rather remarkable properties of Sylow normalizers. Since neither "Sylow subloops" nor normalizers exist, in general, for loops, some of the proofs and counterexamples obtained for loops expose the essential reasons behind the success of the theory for groups.

The paper is divided as follows. In Section 1 we present the conditions to be studied and list some of the implications among them which hold for loops in general. Various pathological examples lead us to restrict ourselves to power-associative loops. We begin Section 2 with a general theorem about power-associative loops. Although the result is not deep, it seems to be new, perhaps because no one needed it before. Using this theorem, we next give a method for constructing new, power-associative loops from old ones. In particular, we construct enough pathological examples to show that the results for power-associative loops must be meager. In Section 3, therefore, we consider diassociative loops. We are able to show that some reasonably interesting implications hold for rather restricted classes of diassociative loops, and we obtain an example which shows that even commutative, diassociative 2-loops are ill-mannered, indeed. In the final section, we touch briefly on the problem for Moufang loops and note that a certain amount of pathology is still present.

1. The problem for general loops

Throughout what follows we use the notation of [1]. In addition, the symbol $\langle A \rangle$ stands for the subloop generated by the set A, and $|A|$ is the order of $\langle A \rangle$. We use the symbol $A \leq B$ ($A < B$) to stand for the statement that A is a subloop of B (and is not B) and use $A \lhd B$ to mean that A is a normal subloop of B. We parallel the definition in Kurosh [5, p. 215] and define an N-loop to be a loop in which every proper subloop is normal in a strictly larger subloop. Finally, if π is a set of primes, we let π' be the complementary set and call a loop a π-loop in case it is power-associative and con-

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