ADJOINT FUNCTORS AND TRIPLES[1]

BY

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A *triple* $\mathbf{F} = (F, \eta, \mu)$ in a category $\mathcal{A}$ consists of a functor $F : \mathcal{A} \to \mathcal{A}$ and morphisms $\eta : 1_{\mathcal{A}} \to F$, $\mu : F^2 \to F$ satisfying some identities (see §2, (T.1)–(T.3)) analogous to those satisfied in a monoid. Cotriples are defined dually.

It has been recognized by Huber [4] that whenever one has a pair of adjoint functors $T : \mathcal{A} \to \mathcal{B}$, $S : \mathcal{B} \to \mathcal{A}$ (see §1), then the functor TS (with appropriate morphisms resulting from the adjointness relation) constitutes a triple in $\mathcal{B}$ and similarly ST yields a cotriple in $\mathcal{A}$.

The main objective of this paper is to show that this relation between adjointness and triples is in some sense reversible. Given a triple $\mathbf{F}$ in $\mathcal{A}$ we define a new category $\mathcal{A}^F$ and adjoint functors $T : \mathcal{A}^F \to \mathcal{A}$, $S : \mathcal{A} \to \mathcal{A}^F$ such that the triple given by TS coincides with $\mathbf{F}$. There may be many adjoint pairs which in this way generate the triple $\mathbf{F}$, but among those there is a universal one (which therefore is in a sense the "best possible one") and for this one the functor T is faithful (Theorem 2.2). This construction can best be illustrated by an example. Let $\mathcal{A}$ be the category of modules over a commutative ring K and let Λ be a K-algebra. The functor $F = \Lambda\otimes$ together with morphisms η and μ resulting from the morphisms $K \to \Lambda$, $\Lambda \otimes \Lambda \to \Lambda$ given by the K-algebra structure of Λ, yield then a triple $\mathbf{F}$ in $\mathcal{A}$. The category $\mathcal{A}^F$ is then precisely the category of Λ-modules. The general construction of $\mathcal{A}^F$ closely resembles this example. As another example, let $\mathcal{A}$ be the category of sets and let F be the functor which to each set A assigns the underlying set of the free group generated by A. There results a triple $\mathbf{F}$ in $\mathcal{A}$ and $\mathcal{A}^F$ is the category of groups.

Let $\mathbf{G} = (\delta, \varepsilon, G)$ be a cotriple in a category A. It has been recognized by Godement [3] and Huber [4], that the iterates G^n of G together with face and degeneracy morphisms

$$G^{n+1} \to G^n, \qquad G^n \to G^{n+1}$$

defined using ε and δ yield a simplicial structure which can be used to define homology and cohomology.

Now if $\mathbf{F}$ is a triple in $\mathcal{A}$, then one has an adjoint pair $T : \mathcal{A}^F \to \mathcal{A}$, $S : \mathcal{A} \to \mathcal{A}^F$ and therefore one has an associated cotriple $\mathbf{G}$ in $\mathcal{A}^F$. This in turn yields a simplicial complex for every object in $\mathcal{A}^F$, thus paving the way for homology and cohomology in $\mathcal{A}^F$. In §4 we show that under suitable

Received April 30, 1964.

[1] The first author was partially supported by a contract from the Office of Naval Research and by a grant from the National Science Foundation while the second author was partially supported by an Air Force contract.