AN ABSTRACT EXTENT FUNCTION[1]

BY

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It is the purpose of this note to define an abstract process, under which, as special cases, will fall the apparently diverse concepts of Lebesgue m-area for mappings from finitely triangulable spaces, the various Lebesgue-Williams areas for mappings from compact metric spaces, outer measures in an abstract set, metric outer measures, such as Hausdorff r-measure, in a metric space, the Daniell-Stone upper integral, the Burkill lower integral for interval functions, and perhaps others.

The form of our definition of the extent function $M(u)$ was suggested by the definitions of m-area given by R. F. Williams [2]. Its substance can be regarded as an extension of the ideas of Lebesgue [6], and of Fréchet [1], who was, apparently, the first to notice that, in the classical case of surface area, Lebesgue's definition may be viewed as a process for extending a semi-continuous function. See also M. H. Stone [4] in connection with what we call measuring systems.

Although our abstract process does not, in general, provide semi-continuous extensions in Fréchet's sense [1], the extent function $M(u)$ which arises is always semi-continuous. It will be clear that while our present definition leads to properties of one-sided lower-semi continuity, the definition may be modified so that, in general, functions exhibiting any of four types of semi-continuity will arise.

Measuring systems and the definition of $M(u)$

A function σ on $U \times U$ to R, where U is a set and R is the set of non-negative real numbers, will be called an *écart* for U and the pair (U, σ) will be called an *écarted space* if σ satisfies the following two conditions:

$$(1) \qquad \sigma(u, u) = 0 \qquad \text{for all } u \in U,$$

$$(2) \qquad \sigma(u, v) \leq \sigma(u, w) + \sigma(w, v) \qquad \text{for all } u, v, w \in U.$$

If U is a set, then a quintuple $\mathfrak{M} = [\sigma, A, q, d, v]$, where σ is an écart for U, A is a set, q is a function on A to U, d is a function on A to R and v is a function on A to R will be called a *measuring system* for U.

For a given measuring system $\mathfrak{M} = [\sigma, A, q, d, v]$ for U, we define, for each $u \in U$, the following subset $R_{\mathfrak{M}}(u)$ of R:

$R_{\mathfrak{M}}(u) = \{r \in R \mid$ for every $\varepsilon > 0$, there exists an $a \in A$ such that $\sigma(u, q(a)) < \varepsilon$, $d(a) < \varepsilon$ and $v(a) < r + \varepsilon\}$.

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