THE LINEAR CUBIC p-ADIC RECURRENCE AND ITS VALUE FUNCTION

BY

HAROLD C. KURTZ

1. Introduction and summary of results

Let

$$\Omega_{n+3} = P\Omega_{n+2} - Q\Omega_{n+1} + R\Omega_n \tag{1.1}$$

be a linear cubic p-adic recurrence with coefficients in the rational p-adic field R_p. The roots α, β, and γ of the characteristic polynomial

$$f(Z) = Z^3 - PZ^2 + QZ - R = (Z - \alpha)(Z - \beta)(Z - \gamma)$$

are p-adic algebraic numbers generating the root field $R_p(\alpha, \beta, \gamma) = \mathfrak{R}_p$ and will be assumed distinct and nonzero.

Let $(W_n) : W_0, W_1, \cdots, W_n, \cdots$ be a solution of (1.1) with given initial values W_0, W_1, and W_2 in R_p not all zero, and let $w_n = \phi(W_n)$ be the p-adic value of W_n. We investigate the following "valuation problem": Given a sequence (W_n) satisfying (1.1) with specified initial values as above, to determine $\phi(W_n)$. This problem is trivial if one of the ratios of the roots of $f(Z)$ is a root of unity in $\mathfrak{R}_p$; $f(z)$ is then termed degenerate. Hence we assume nondegeneracy, i.e., $(\alpha/\beta)^n$, $(\beta/\gamma)^n$, and $(\alpha/\gamma)^n \not\equiv 1$ for all positive integers n.

We show that we may restrict ourselves to recurrences whose coefficients and initial values are p-adic integers where at least one coefficient and one initial value are p-adic units. Except when $p = 2$ or 3, we need only consider these cases:

I P, Q, and R all p-adic units,
II P and Q units, R a non-unit,
III P a unit, Q and R non-units.

In Case III, the determination of $\phi(W_n)$ is trivial; for $n \geqq$ some n_0, $\phi(W_n)$ equals a constant. In II, the Hensel Lemma enables us to analyze the valuation of the cubic recurrence in terms of the valuation of the quadratic recurrence (results (3.2)–(3.5)), explicit formulas for the latter being given in Ward's paper [2]. Case I has been studied by Ward [1] when coefficients and initial values are rational integers; the results are extended to recurrences where these are p-adic integers in Section 4.

It appears likely that for a given integer t, the valuation problem for the t^{th} order nondegenerate recurrence

$$\Omega_{n+t} = A\Omega_{n+(t-1)} + \cdots + M\Omega_n$$

Received August 24, 1962.