A PRIORI ESTIMATES FOR DIFFERENTIAL OPERATORS IN L_∞ NORM[1]

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It is well known that if A and B are constant-coefficient partial differential operators, with A elliptic and order $B \leqq$ order A, then

$$\int | B\varphi |^2 \leqq \text{const} \int (| \varphi |^2 + | A\varphi |^2)$$

for all infinitely differentiable functions φ of compact support. The proof of this "a priori estimate" uses Fourier transforms and the Plancherel theorem. Similar estimates are known for p^{th} powers $(1 < p < \infty)$ in place of squares, although the easy proof for $p = 2$ does not generalize. In the present paper we investigate the limiting case $p = \infty$, where supremum norms appear in place of L_p integral norms. This case turns out to be genuinely exceptional. For instance (Proposition 2) if A and B have the same order and $B \neq cA$, then no a priori estimate

$$\sup | B\varphi | \leqq \text{const} \sup (| \varphi | + | A\varphi |)$$

is possible. But (Proposition 5) if B has strictly lower order, and A is elliptic, then the estimate is reinstated. In fact (converse half of Proposition 5) in dimension $n \geqq 3$ this property is characteristic of elliptic operators, just as the L_2 a priori estimate is characteristic of elliptic operators for the case of equal orders. Before proving these last assertions we must establish (Propositions 3 and 4) some basic facts about the n-dimensional Fourier transform that do not seem to be in the literature. The connection between a priori estimates and Fourier transforms is explained in Proposition 1.

The other limiting case $p = 1$ has recently been treated by Ornstein [4]. The results for L_1 are essentially the same as those for L_∞, but seem to be much harder to prove.

Operator domination

If

$$A = \sum a_e \left(\frac{\partial}{\partial x}\right)^e = \sum a_{e_1 \cdots e_n} \left(\frac{\partial}{\partial x_1}\right)^{e_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{e_n}$$

is a partial differential operator with constant coefficients, then its *full characteristic polynomial* is

$$P = \sum a_e (ix)^e = \sum a_{e_1 \cdots e_n} (ix_1)^{e_1} \cdots (ix_n)^{e_n}.$$

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