

CONVOLUTIONS, MEANS, AND SPECTRA

BY
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1. Introduction

In two recent papers ([6] and [7] of the bibliography) Kesten studied symmetric probability densities φ on discrete groups G . For each such φ he defined first a matrix and then an operation on $l_2(G)$, an operation easily shown to be equivalent to right convolution, $\circ\varphi$, by φ . The properties found in [6] for the family of groups where $\lambda_2(\varphi)$, the spectral radius of $\circ\varphi$ operating in $l_2(G)$, is 1 suggested to Kesten the result he proved in [7]: $\lambda_2(\varphi) = 1$ for every symmetric φ on G if and only if there is an invariant mean on the bounded functions of G .

In this paper I exploit some results of my earlier work, [1] on invariant means and [2] on uniform rotundity, to give a simpler proof of a more general result. My proof uses uniform rotundity in place of symmetry and strong amenability (see [1, §5, Theorem 1]) to replace Følner's characterization [4] of groups with invariant means. This simpler proof with no dependence on self-adjointness applies to all l_p spaces, $p > 1$, and applies also to some semigroups which are not groups. In this semigroup case, where right and left invariance are independent properties of means, it turns out that right invariance of means is to be used in studying right convolutions.

The relation between these results and those of the paper [3] of Dieudonné, which is concerned with locally compact groups with the property that right convolution by each (Haar measurable) probability density is an operator of norm one in every L_p over G , are discussed but not settled in the final section of this paper.

Theorem 3 isolates from the many results a response to Kesten's hope that some more direct construction of invariant means might be found when 1 is in the spectrum of enough operators $\circ\varphi$.

2. Principal results

It is assumed in this section that each semigroup Σ discussed has right cancellation (rc) and has a nonempty set U of right units (ru). If φ is a probability density on Σ , $P_\varphi = \{\sigma : \varphi(\sigma) > 0\}$. $\circ\varphi$, or right convolution by φ , is defined precisely in the next section, as are $\delta\sigma$ and f^φ .

THEOREM 1. *The following conditions on an (rcru) semigroup Σ are equivalent:*

- (a) Σ is right amenable.

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