SOME AREA THEOREMS AND A SPECIAL COEFFICIENT THEOREM

BY
JAMES A. JENKINS[1]

1. The first significant method used in the theory of univalent functions was the area principle. Its use led to the initial results on sharp bounds in the simple standard results for univalent functions. In the more modern developments of the theory the chief tools have been Löwner's parametric method, the method of the extremal metric, the method of contour integration, and the variational method. These methods have been employed to deal with a wide range of results, and frequently a given result has been obtained separately by the use of several or even all of them. Among them the method of the extremal metric and the method of contour integration share with the area principle the feature that an essential step in the procedure is the assertion that the integral of a positive function is nonnegative. The method of contour integration was first used by Grunsky [3]. He also used it to obtain some quite general relationships for the coefficients of univalent functions [4]. A closely related method has been used by Nehari [10]. The method of contour integration has also been used by Golusin, Schiffer and Spencer [13], and others.

In this paper we will observe that many of the results obtained by the method of contour integration, including Grunsky's coefficient inequalities, can be obtained by a direct application of the area principle. Indeed in all these cases the area principle provides a sharper inequality.

On the other hand many of these applications of the area principle are consequences of a general result which we obtain by the method of the extremal metric. Moreover there are general circumstances in which the latter applies while the former does not.

2. We will begin by recalling some notations and known results.

Let D be a domain of finite connectivity in the z-sphere containing the point at infinity, and let $\Sigma(D)$ be the class of functions $f(z)$ univalent in D, regular apart from a simple pole at the point at infinity where the Laurent expansion is given by

$$f(z) = z + c_0 + c_1/z + \cdots + c_n/z^n + \cdots .$$

Let $\Sigma'(D)$ denote the subclass of $\Sigma(D)$ of functions for which $c_0 = 0$. If D is the domain $|z| > 1$, we denote these classes simply by Σ and Σ'. Without loss of generality we can always assume D to be bounded by analytic curves.

Received July 30, 1962.

[1] This research was supported by the Air Force Office of Scientific Research.