ARITHMETIC PROBLEMS CONCERNING CAUCHY'S FUNCTIONAL EQUATION[1]

BY

Ch. Pisot and I. J. Schoenberg

Introduction

The present study concerns some modifications of the functional equation

$$f(x + y) = f(x) + f(y)$$

which arose in connection with certain problems on additive arithmetic functions. An arithmetic function $F(n)$ $(n = 1, 2, \cdots)$ is said to be *additive* provided that $F(mn) = F(m) + F(n)$ whenever $(m, n) = 1$. In [2] Erdös found that if the additive function $F(n)$ is nondecreasing, i.e., $F(n) \leqq F(n + 1)$ for all n, then it must be of the form $F(n) = C \log n$. This result was rediscovered by Moser and Lambek [3], and recently further proofs were given by Schoenberg [4] and Besicovitch [1].

Erdös' remarkable characterization of the function $\log n$ raises the following question: Let $p_1, p_2, \cdots, p_k$ be a given set of k $(\geqq 2)$ distinct prime numbers. Let $F(n)$ be defined on the set A of integers n which allow no prime divisors except those among $p_1, \cdots, p_k$, and let $F(n)$ be additive, i.e.,

$$F(p_1^{u_1} p_2^{u_2} \cdots p_k^{u_k}) = F(p_1^{u_1}) + F(p_2^{u_2}) + \cdots + F(p_k^{u_k}). \tag{1}$$

If we assume $F(n)$ to be nondecreasing over the set A, is it still true that $F(n) = C \log n$?

One of us having communicated this question to Erdös, received in reply a letter dated February 13, 1961, in which Erdös states, with brief indications of proofs, that the answer to the above question is affirmative if $k = 3$ and negative if $k = 2$. We shall deal with these results below under more general assumptions. The negative answer for $k = 2$ is already established by any counterexample, a particularly simple one being

$$F(n) = [\log n/\log p_1] + [\log n/\log p_2]. \tag{2}$$

Indeed, it is easy to verify that this particular monotone $F(n)$ satisfies (1), for $k = 2$, while it is not of the form $C \cdot \log n$, for $n = p_1^{u_1} p_2^{u_2}$ (see also Section 12).

At this point we change notations. If we write $F(e^x) = f(x)$, $\log p_i = \alpha_i$, the relation (1) becomes

$$f(u_1 \alpha_1 + \cdots + u_k \alpha_k) = f(u_1 \alpha_1) + \cdots + f(u_k \alpha_k) \quad (u_i \text{ integers} \geqq 0). \tag{3}$$

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