LOCAL TIMES FOR A CLASS OF MARKOFF PROCESSES

BY

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Let $X(\tau, \omega)$ be a one-dimensional Markoff process defined on a probability space $(\Omega, \mathfrak{F}, P)$. For any positive real t and any $\omega \in \Omega$ we can define a measure, $\mu(\cdot, t, \omega)$, on R, the real numbers, by

$$\mu(A, t, \omega) = \text{LM}\{\tau : X(\tau, \omega) \in A, 0 \leqq \tau < t\}, \tag{0.1}$$

where LM represents Lebesgue measure. Trotter [3] showed that if $X(\tau, \omega)$ is Brownian motion, then, for almost all ω, $\mu(\cdot, t, \omega)$ has a continuous density function, i.e., there exists a function $L(x, t, \omega)$, defined for all $x \in R$ and all positive t, continuous jointly in x and t, such that

$$\mu(A, t, \omega) = \int_A L(x, t, \omega)\, dx \tag{0.2}$$

for every Borel set A. $L(x, t, \omega)$ is called the "local time" at x up to time t. In this paper we investigate the following problem: For θ a given Borel measure on R, when will $\mu(\cdot, t, \omega)$ have a continuous θ-density, i.e., when will there exist, for almost all ω, a function $L(x, t, \omega)$, defined for all $x \in R$ and all positive t, continuous jointly in x and t, such that

$$\mu(A, t, \omega) = \int_A L(x, t, \omega)\theta\{dx\} \tag{0.3}$$

for every Borel set A? We shall show that such an $L(x, t, \omega)$ will exist, for almost all ω, whenever the transition probabilities of $X(\tau, \omega)$ satisfy certain conditions (involving θ). In particular, we shall show that if $X(\tau, \omega)$ is a stable process of index α, $1 < \alpha \leqq 2$, then, for almost all ω, there will exist a function $L(x, t, \omega)$ satisfying (0.2). Since Brownian motion is a stable process of index 2, this offers a new proof of Trotter's result.

1. Preliminary material

The purpose of this section is to explain briefly certain concepts arising in the theory of Markoff processes which will be used in later sections.

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