HOMOGENEITY AND BOUNDED ISOMETRIES IN MANIFOLDS OF NEGATIVE CURVATURE

BY

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1. Statement of results

The object of this paper is to prove Theorem 3 below, which gives a fairly complete analysis of the structure of Riemannian homogeneous manifolds of nonpositive sectional curvature. The main tool is

THEOREM 1. *Let M be a complete connected simply connected Riemannian manifold with every sectional curvature nonpositive. Let γ be an isometry of M; given $m \in M$, let X_m be the (unique by hypothesis on M) tangentvector to M at m such that* $\exp(X_m) = \gamma(m)$; *let X be the vectorfield on M defined by the X_m. Let M_0 be the Euclidean factor in the de Rham decomposition of M, so $M = M_0 \times M'$ where M' is the product of the irreducible factors. Then these are equivalent*:

(1) *There is an ordinary translation γ_0 of the Euclidean space M_0 such that the action of γ on $M = M_0 \times M'$ is given by $(m_0, m') \to (\gamma_0 m_0, m')$.*

(2) *X is a parallel vectorfield on M.*

(3) *γ is a Clifford translation[2] of M.*

(4) *γ is a bounded isometry[3] of M.*

In particular, if M_0 is trivial, then every bounded isometry of M is trivial.

As an immediate consequence of Theorem 1, we have

THEOREM 2. *Let M be a complete connected simply connected Riemannian manifold of nonpositive sectional curvature, and let Γ be a properly discontinuous group of fixed-point-free isometries of M. Then these are equivalent*:

(1) *M/Γ is isometric to the product of a flat torus with a complete simply connected Riemannian manifold of nonpositive sectional curvature.*

(2) *Every element of Γ is a Clifford translation of M.*

(3) *Every element of Γ is a bounded isometry of M.*

In particular, if one of these conditions holds, then M/Γ is diffeomorphic to the product of a torus and a Euclidean space.

To prove the following theorem, which is our goal, one notes that (1)

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[1] This research was done at the N. S. F. Summer Institute in Relativity and Differential Geometry, University of California at Santa Barbara, 1962.

[2] An isometry of a metric space is called a *Clifford translation* if the distance between a point and its image is the same for every point.

[3] A *bounded isometry* of a metric space is an isometry such that the distance between a point and its image is at most equal to some bound.