CONJUGACY IN THE REAL THREE-DIMENSIONAL ORTHOGONAL GROUPS

BY

BARTH POLLAK

Introduction

There are two three-dimensional orthogonal groups over the field $\mathbf{R}$ of real numbers determined by whether or not the quadratic form which defines the "metric" on the space is anisotropic (ordinary Euclidean 3-space) or not. In both cases the commutator subgroup $\mathbf{\Omega}$ is a simple group. (When the space is anisotropic, the commutator subgroup is the set of all isometries of determinant $+1$; when the space is isotropic, it is a normal subgroup of index 2 in that group.) Thus if $a \neq 1$ is in $\mathbf{\Omega}$ and $b \in \mathbf{\Omega}$, there exists a positive integer n such that

$$(*) \qquad b = \textstyle\prod_{i=1}^{n} t_i\, a^{\pm 1} t_i^{-1}.$$

Let $\mathbf{N}_a(b)$ denote the smallest n for which $(*)$ is true. By the use of quaternions, we give an explicit formula for $\mathbf{N}_a(b)$ in both cases.

1. Quaternion algebras

Let K be a field of characteristic $\neq 2$. By a *quaternion algebra* $\mathbf{H}$ over K we mean a central simple associative algebra of dimension 4 over K. It is well known that $\mathbf{H}$ has a basis of the form $1, I, J, IJ$ with 1 the multiplicative identity, $I^2 = \alpha$, $J^2 = \beta$, $IJ = -JI$, where $\alpha, \beta \in K^*$ (the multiplicative group of nonzero elements of K). (See [1, Theorem 27, p. 146].) We shall use the notation (α, β) for a quaternion algebra possessing such a basis. $\mathbf{H}$ possesses an antiautomorphism of period 2 called *conjugation*, the image of $X \in \mathbf{H}$ being denoted X^c. Then we have $X + X^c = S(X)1$, $XX^c = N(X)1$ with $S(X), N(X) \in K$ called respectively the *trace* and *norm* of X, and $X^2 - S(X)X + N(X)1 = 0$ for each $X \in \mathbf{H}$. If $S(X) = 0$, we call X *pure*. If $X = \xi_0 1 + \xi_1 I + \xi_2 J + \xi_3 IJ$, we have

$$(1) \qquad N(X) = \xi_0^2 - \alpha\xi_1^2 - \beta\xi_2^2 + \alpha\beta\xi_3^2\,.$$

For future use we set $\mathbf{H}_1 = \{X \in \mathbf{H} \mid N(X) = 1\}$. We conclude this section by stating

THEOREM 1. *Let $A, B \in \mathbf{H}$. There exists $T \in \mathbf{H}$ such that $B = TAT^{-1}$ if and only if $N(A) = N(B)$ and $S(A) = S(B)$. There exists $T \in \mathbf{H}_1$ such that $B = TAT^{-1}$ if and only if in addition to the above conditions,*

(i) *$(N(B - A^c), S^2(A) - 4N(A)) \cong \mathbf{M}_2(K)$, the algebra of all 2×2 matrices over K provided $N(B - A^c)$ and $S^2(A) - 4N(A)$ are both nonzero;*

(ii) *if $S^2(A) - 4N(A) = 0$, then $N(B - A^c) \in K^2$.*

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