A REMARK ON A PAPER OF MINE ON POLYNOMIALS

BY
KURT MAHLER

1. The measure $M(f)$ of a polynomial

$$f(x) = a_0 x^m + a_1 x^{m-1} + \cdots + a_m$$

with real or complex coefficients is defined by

$$M(f) = \exp\left\{\int_0^1 \log|f(e^{2\pi i t})|\,dt\right\} \quad \text{if} \quad f(x) \not\equiv 0,$$
$$= 0 \quad \text{if} \quad f(x) \equiv 0.$$

Denote by S_{mn} the set of all polynomial vectors

$$\mathbf{f}(x) = (f_1(x), \cdots, f_n(x))$$

with components $f_h(x)$ that are polynomials at most of degree m and that do not all vanish identically. Further put

$$M(\mathbf{f}) = \sum_{h=1}^n M(f_h), \qquad N(\mathbf{f}) = \sum_{h=1}^n \sum_{k=1}^n M(f_h - f_k),$$
$$Q(\mathbf{f}) = N(\mathbf{f})/M(\mathbf{f}).$$

In my paper *On Two Extremum Properties of Polynomials*[1] I proved that the least upper bound

$$C_{mn} = \sup_{\mathbf{f} \in S_{mn}} Q(\mathbf{f})$$

satisfies the nearly trivial inequality

$$(1) \qquad C_{mn} \leqq 2^{m+1}(n-1)$$

and is attained for a polynomial vector

$$\mathbf{F}(x) = (F_1(x), \cdots, F_n(x))$$

in S_{mn} with the following properties:

(2) *Those components $F_h(x)$ of $\mathbf{F}(x)$ that do not vanish identically all have the exact degree m, and all their zeros lie on the unit circle.*

It does not seem to be easy to determine the exact value of C_{mn}. In this note I shall replace (1) by an inequality (9) which is slightly better when m is large relative to $\log n$.

2. Let $\mathbf{F}(x)$ be defined as before. Without loss of generality,

$$F_1(x) \not\equiv 0, \quad \cdots, \quad F_p(x) \not\equiv 0, \qquad F_{p+1}(x) \equiv \cdots \equiv F_n(x) \equiv 0,$$

Received October 7, 1963.

[1] Illinois Journal of Mathematics, vol. 7 (1963), pp. 681–701.