LOCAL BOUNDARY BEHAVIOR OF HARMONIC FUNCTIONS

BY
GILBERT WALTER

1. Introduction

The solution to the Dirichlet poblem on the unit disk, that is, the problem of finding a harmonic function $f(r, \theta)$ in the interior of the disk corresponding to a given function $f(\theta)$ on the boundary, has become part of the folk knowledge of mathematics. It is common knowledge also that $\lim_{r\to 1} f(r, \theta) = f(\theta)$ at each point of continuity of f. The solution to the converse problem, that of finding a boundary function (or some generalization of function) corresponding to a given harmonic function in the interior, is not so well known, but nevertheless has been extensively studied in the last decade. A solution always exists in the space of hyperfunctions H' on the boundary. In fact, these hyperfunctions are exactly the objects giving a solution to the converse problem. Moreover, the original Dirichlet problem has a unique solution when f is a hyperfunction instead of a point function. However, the statement about limits at points of continuity has no meaning for f in H'. It is the purpose of this report to give it meaning and to prove this theorem for f in H'.

Hyperfunctions have been characterized in a number of different ways. Two of them are as equivalence classes of pairs of holomorphic functions and as continuous linear functionals on a space of holomorphic test functions. See e.g. Sato [1], Köthe [2], [3], Lions and Magenes [4], and Schapira [5]. The former characterization enables one to consider them as types of generalized boundary values of harmonic functions and the latter as generalized functions in the sense of Gelfand-Shilov [6]. On the boundary Γ of the unit disk hyperfunctions correspond to exponential trigonometric series $\sum C_n e^{in\theta}$ whose coefficients satisfy

$$\limsup |C_n|^{1/|n|} \leq 1.$$

Thus the space H' contains all distributions on Γ (whose coefficients satisfy $C_n = O(|n|^p))$ and is contained in the space Z' of ultradistributions (since every trigonometric series, no matter what its coefficients are, converges in Z'). See [11].

In the case of distributions, there is an already available concept which corresponds to continuity at a point of a continuous function. It is the concept of point value introduced by Lojasiewicz [9]. A recent characterization of the elements of H' by Johnson [7] as series of distributions allows this concept to be extended in a natural way to hyperfunctions. It is then possible

Received April 27, 1970.