A DECOMPOSITION PROOF THAT THE DOUBLE SUSPENSION OF A HOMOTOPY 3-CELL IS A TOPOLOGICAL 5-CELL[1]

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1. Introduction and definitions

In [5], the author proved that if H^3 is a PL homotopy 3-sphere bounding a compact contractible PL 4-manifold, then the double suspension of H^3 is topologically homeomorphic to the 5-sphere S^5. (We write this as $\Sigma^2 H^3 \approx S^5$, where Σ^2 denotes double suspension and $\approx$ means topologically homeomorphic.) In [10], Siebenmann gives an elegant proof that $\Sigma^2 H^3 \approx S^5$, for any homotopy 3-sphere H^3. However, this proof is somewhat unsatisfactory in that it has to make use of some deep results of the Kirby-Siebenmann triangulation theory, and a key theorem needed to obtain this result was given merely by a reference to a paper by Kirby and Siebermann, which apparently was not even in preprint form at the time.[3] In [6], the author made use of a completely geometrical, but quite involved, argument, outlined to him by Kirby, to show that if F^3 is a homotopy 3-cell, then $\Sigma^2 F^3 \approx I^5$. This requires a long and complicated argument, which depends quite heavily on the full work of [4]. In an addendum to [10], Siebenmann remarks that the same proof used to show that $\Sigma^2 H^3 \approx S^5$, also works to show that $\Sigma^2 F^3 \approx I^5$.

Here, we give an easy decomposition proof that $\Sigma^2 F^3 \approx I^5$, for any homotopy 3-cell F^3. The proof only requires a simple application of the engulfing lemma of [11], plus the fact that all homotopy 3-cells can be triangulated [1] and some basic fundamentals of geometric PL theory. Moreover, by using the collaring theorem of [2] and the topological h-cobordism theory of [3] (which itself only requires [2] and the engulfing lemma), the proof given here actually can be used to show that $\Sigma^2 F^3 \approx I^5$, without even using the fact that 3-manifolds can be triangulated (also refer to the remarks at the beginning of §5).

In Corollary 4.3, we show that if M^3 is an arbitrary homotopy 3-sphere and $h : S^2 \to N^2 \subset M^3$ is a homeomorphism carrying S^2 onto the locally flat submanifold N^2 of M^3, then there exists a homeomorphism

$$H : (\Sigma^2(v_1 * S^2 * v_2), \Sigma^2 S^2) \to (\Sigma^2 M^3, \Sigma^2 N^2)$$

such that $H \mid \Sigma^2 S^2 = \Sigma^2 h$ (here $*$ denotes join and $\Sigma^2 h : \Sigma^2 S^2 \to \Sigma^2 N^2$ denotes

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[3] After this paper was written, Siebenmann informed the author that he and Kirby also know an "elementary" proof of this result using engulfing and an infinite meshing process of Černavskiĭ; however, this also is not written down anywhere.