ABSTRACT HOMOTOPY IN CATEGORIES OF FIBRATIONS AND THE SPECTRAL SEQUENCE OF EILENBERG AND MOORE

BY

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In previous articles [5], [6] we have examined those formal characteristics of the category of topological spaces which make possible the constructions of homotopy theory and, accordingly, of homology theory. These characteristics are shared by many other categories, which thus admit their own homotopy and homology theories: as to the latter we might say that, as "extraordinary" homology generalizes by changing coefficient objects, so a further generalization occurs in changing the domain of homology theory.

Our primary purpose here is to exhibit the existence of such structure in three cases: the category of spaces over a fixed space; the category of Hurewicz fibrations over a fixed space; the category of spaces provided with a fixed group of operators.[2] We thus justify assertions made in the references cited above. This is not however an empty generalization. We shall use it to give perspicuous derivations of two spectral sequences due to Eilenberg and Moore [4]. One of these involves the homology of the pullback of a fibration, the other the homology of the fiber bundle with a prescribed fiber associated to a principal bundle.

These spectral sequences have been derived in a number of ways. The original techniques of Eilenberg and Moore use chain-complex arguments and appear to give results only for singular homology. Rector [8] derives the pullback spectral sequence by cosimplicial methods which seem to be quite different from those advanced here. The construction of L. Smith [10], however, is quite similar to the one below, as is that of Steenrod and Rothenberg [8] for the associated bundle.

Smith (loc. cit.) remarks on the analogy between the pullback spectral sequence and the Adams spectral sequence (cf. [1]). There seems to be no doubt that they belong to a common domain whose boundaries however have not yet been completely determined. The argument below, which uses stable homotopy methods and is couched in terms of homology rather than cohomology (thus avoiding finiteness restrictions after the fashion of Eilenberg and Moore) reinforces this analogy.

In §§1, 2 below we restate the axioms for abstract homotopy theory in an

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[2] I understand that J. F. McClendon and I. M. James have investigated these categories, or near neighbors, from similar points of view in as yet unpublished studies.

Added in proof. Cf. also I. M. James, *Ex-homotopy theory I*, Illinois J. Math., vol. 15 (1971), pp. 324–337.