INJECTIVES AND HOMOTOPY[1]

BY

J. A. SEEBACH, JR.

Any functor P on a category $\mathcal{A}$ determines an equivalence relation on the morphisms of $\mathcal{A}$ which is compatible with composition in $\mathcal{A}$. We call any relation determined by a functor a *homotopy* and develop the ideas of cylinder and cone functors in this general context. An appropriate generalization of the homotopy extension axiom implies that a cone functor is essentially a functorial injective for the category. This structure occurs in the cases of the usual homotopy on CW-complexes and the Eckmann-Hilton injective homotopy of modules which we present in a generalized categorical context.

For any category $\mathcal{A}$ together with a class of its morphisms M, the projection functor $P : \mathcal{A} \to \mathcal{A}/M$ yields a homotopy, where $\mathcal{A}/M$ is the Gabriel-Zisman category of fractions of $\mathcal{A}$ by M. Indeed for the "right" choice of M, P yields the well-known homotopies above: to be precise, take M to be the class of coretracts $i : X \to Y$ for all X, Y such that $Y/i(X)$ is injective. This enables us to determine the "right" homotopy from the knowedge of the contractible objects (injectives) alone.

1. Categorical preliminaries

If $\mathcal{A}$ is a category, a *homotopy* (or congruence) on $\mathcal{A}$ is an equivalence relation $\sim$ on each of the sets $\mathcal{A}(X, Y)$ of morphisms between objects of $\mathcal{A}$ which is compatible with composition; that is, $f \sim g$ implies $fh \sim gh$ and $kf \sim kg$ for all h, k for which the compositions are defined. If $\mathcal{A}$ has a homotopy $\sim$ the *homotopy category* of $\mathcal{A}$ with respect to $\sim$ is the category $\mathcal{A}/\sim$ whose objects are those of $\mathcal{A}$ and whose morphisms are the equivalence classes under $\sim$ of $\mathcal{A}(X, Y)$ together with the projection functor $\rho : \mathcal{A} \to \mathcal{A}/\sim$ which is the identity on objects and maps each morphism f to its equivalence class $[f]$ under $\sim$. The functor ρ determined by the homotopy is clearly universal with respect to all functors $F : \mathcal{A} \to \mathcal{B}$ such that $f \sim g$ implies $F(f) = F(g)$. Conversely any functor $F : \mathcal{A} \to \mathcal{B}$ defines a homotopy by $f \sim g$ iff $F(f) = F(g)$ and f, g have the same domain and codomain, though $\mathcal{B}$ need not then be $\mathcal{A}/\sim$, e.g. when F is not one-to-one on objects.

Three examples indicate the applicability and generality of these techniques. These are the usual homotopy of continuous functions on CW-complexes, the fibre homotopy in the category of functions to a fixed base space, and the Eckmann-Hilton injective homotopy of modules of which a development is given in Section 4. For more details see Hilton [4].

Received April 10, 1970.

[1] This research was supported in part by grants from the Research Corporation and the ETH.