ILLINOIS JOURNAL OF MATHEMATICS
Volume 27, Number 1, Spring 1983

DIOPHANTINE APPROXIMATION AND NORM DISTRIBUTION IN GALOIS ORBITS

BY

C. J. BUSHNELL

This paper investigates another facet of a problem, introduced in [1] and [2], concerning relations (or lack of them) between arithmetic properties and Galois-module properties of algebraic numbers. We let K be an algebraic number field, normal (and of finite degree) over the field $\mathbf{Q}$ of rational numbers, and we write $\Gamma = \text{Gal}(K/\mathbf{Q})$. The natural action $\gamma : b \mapsto b^\gamma$, $\gamma \in \Gamma$, $b \in K$, of the Galois group on K extends to a module structure over the rational group algebra $\mathbf{Q}\Gamma$. Explicitly, if $a \in K$, and if

$$x = \sum_{\gamma \in \Gamma} x_\gamma \gamma, \quad x_\gamma \in \mathbf{Q},$$

is a typical element of $\mathbf{Q}\Gamma$, the module action is given by the formula

$$a \cdot x = \sum_{\gamma \in \Gamma} a^\gamma x_\gamma.$$

One knows (Hilbert's Normal Basis Theorem) that K is a free $\mathbf{Q}\Gamma$-module of rank one. That is, there exists $a \in K$ such that $K = a \cdot \mathbf{Q}\Gamma$, or, equivalently, the conjugates a^γ, $\gamma \in \Gamma$, of a are linearly independent over $\mathbf{Q}$.

This module structure naturally leads to others. Most notably, the ring $\mathfrak{o}_K$ of algebraic integers in K is a module over the integral group ring $\mathbf{Z}\Gamma$. If every prime of $\mathbf{Q}$ is at most tamely ramified in K, $\mathfrak{o}_K$ is "usually" a free $\mathbf{Z}\Gamma$-module: there exists $a \in K$ such that $\mathfrak{o}_K = a \cdot \mathbf{Z}\Gamma$. (This holds if, for example, Γ has no irreducible symplectic characters. See [4] for a complete account.) It is the arithmetic properties of these elements a with $a \cdot \mathbf{Z}\Gamma = \mathfrak{o}_K$ which primarily interest us. Here we are concerned with their norms.

This is better considered in a more general context. We fix an element $a \in K$ such that $a \cdot \mathbf{Q}\Gamma = K$. (These elements are, in some geometrical sense, typical.) The linear isomorphism $\mathbf{Q}\Gamma \simeq K$ given by $x \mapsto a \cdot x$, $x \in \mathbf{Q}\Gamma$, enables us to transfer arithmetical functions from K to $\mathbf{Q}\Gamma$. In particular, we write

$$\nu_a(x) = N_{K/\mathbf{Q}}(a \cdot x), \quad x \in \mathbf{Q}\Gamma,$$

Received March 27, 1981.