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EULER CHARACTERISTICS OVER UNRAMIFIED REGULAR LOCAL RINGS

BY

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Let M, N be finitely generated modules over a local ring (R, m) (all rings are assumed commutative, with identity; (R, m) is "local" means that R is Noetherian with maximal ideal m). If $Tor_j^R(M, N)$ has finite length for $j \geqslant i$, i a nonnegative integer, and vanishes for all sufficiently large j, we define

$$\chi_i^R(M, N) = \sum_{j \geqslant i} (-1)^{j-i} l(Tor_j^R(M, N)),$$

where l denotes length. The main result here is the following:

THEOREM. *Let R be an unramified regular local ring and let M, N be finitely generated R-modules such that $Tor_i^R(M, N)$ has finite length, $i \geqslant 1$. If $\chi_i(M, N) = 0$, then*

$$Tor_j^R(M, N) = 0 \quad \text{for } j \geqslant i.$$

It was already known (see [1], [2], [3]) that if R is regular and $Tor_i^R(M, N)$ is 0 (respectively, has finite length) then

$$Tor_j^R(M, N) = 0$$

(respectively, has finite length) for $j \geqslant i$. Moreover, in [3] it is shown that if R is an unramified regular local ring and $Tor_i^R(M, N)$ has finite length, $i \geqslant 1$, then $\chi_i^R(M, N) \geqslant 0$, and that if $i \geqslant 2$ or M or N is torsion-free, then $\chi_i^R(M, N) = 0$ if and only if $Tor_j^R(M, N) = 0, j \geqslant i$. Thus, the theorem is new only in the case $i = 1$.

As usual, we may reduce at once to the case where R is complete and then assume $R = V[[x_2, \ldots, x_n]]$, where $n = \dim R$ and V is a complete discrete valuation ring with maximal ideal $x_1 V$. We abbreviate $x = x_1$.

We write $M \hat{\otimes}_V N$ and $\hat{Tor}_i^V(M, N)$ for the complete tensor product and complete Tor_i, respectively, of M and N over V (see [4, p. V-6].) Let $S = R \hat{\otimes}_V R$. S is regular and if we map $S \to R$ by

$$a \hat{\otimes} b \mapsto ab,$$

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