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THE LOOP SPACE OF THE Q-CONSTRUCTION[1]

BY

HENRI GILLET AND DANIEL R. GRAYSON

The higher algebraic K-groups are defined as $K_i\mathcal{M} = \pi_{i+1}|Q\mathcal{M}| = \pi_i\Omega|Q\mathcal{M}|$ for an exact category $\mathcal{M}$. We present a simplicial set $G\mathcal{M}$ with the property that $|G\mathcal{M}|$ is naturally homotopy equivalent to the loop space $\Omega|Q\mathcal{M}|$, and thus $K_i\mathcal{M} = \pi_i|G\mathcal{M}|$. In this way we given an algebraic description of the loop space, which a priori, has no such description.

The case where $\mathcal{M}$ is a category in which all the exact sequences split was done by Quillen with his category $S^{-1}S$. In fact, the definition of our space $G\mathcal{M}$ is a simple generalization of the definition of $S^{-1}S$. Its vertices are all pairs (M, N) of objects of $\mathcal{M}$, and its edges connecting (M, N) to (M', N') are all pairs of exact sequences

$$0 \to M \to M' \to C \to 0, \quad 0 \to N \to N' \to C \to 0.$$

Higher dimensional simplices are defined analogously. There is an isomorphism $\pi_0 G\mathcal{M} \cong K_0\mathcal{M}$, with (M, N) corresponding to $[M] - [N]$.

The simplicial techniques (Section 1) used in the proof that $|G\mathcal{M}| \sim \Omega|Q\mathcal{M}|$ are generalizations of Theorems A and B of Quillen [5]; they apply to simplicial sets rather than just to categories. There is a canonical procedure (subdivision) for converting simplicial sets to categories, but our techniques are not based on this.

The main idea from Quillen's proof of the statement $S^{-1}S = \Omega|Q\mathcal{M}|$ also appears here, but in a more understandable guise. The motto might be "use algebra to add, and topology to subtract". This can be explained briefly by considering supermodules $M \supset N$ of an R-module N (i.e., injections from N into another module). Using algebra, we may "add" them thus: $M_1 + M_2 := M_1 \amalg_N M_2$. We have the equation $M + M = M + (N \oplus M/N)$, and it turns out that by topology (i.e., in the homotopy groups) subtraction is allowed, and yields the equation $M = N \oplus M/N$.

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