ILLINOIS JOURNAL OF MATHEMATICS
Volume 41, Number 2, Summer 1997

CANONICAL RING OF A CURVE IS KOSZUL: A SIMPLE PROOF

GIUSEPPE PARESCHI AND B. P. PURNAPRAJNA

1. Introduction

In this article we prove, for canonical model of curves, a theorem illustrating the general principle that (to paraphrase Arnold) any homogeneous ring that has a serious reason for being quadratically presented is *Koszul*. In this case we give a new proof, which is both elementary and geometric, of a theorem of Finkelberg and Vishik [VF] (see also [Po]) which says that whenever the canonical ring of a smooth complex projective curve is quadratically presented, it is *Koszul*. Our method is different from [Po]. We use vector bundle technique, building upon the one used in [GL]. We would also like to mention here that our methods fit a more general principle as shown in [GP1], [GP2] and [GP3].

A. The Koszul conditions. Let k be a field. A (commutative) graded k-algebra of the form $R := k \oplus R_1 \oplus \cdots \oplus R_n \cdots$ is said to be *Koszul* if its Koszul complex is exact, or, equivalently, if $k = R/R_{>0}$ has a *linear* minimal resolution over R; namely

$$\cdots \to E_p \to E_{p-1} \to \cdots \to E_2 \to E_1 \to E_0 \to k \to 0$$

with $E_0 = R$ and $E_p = R(-p)^{\oplus r(p)}$ for any $p \geq 1$. Denote the syzygy modules by $R^{(p)} := \ker(E_p \to E_{p-1})$; this means that for any $p \geq 0$ the $R^{(p)}$'s are generated in degree $p+1$ (the minimal degree) as graded R-modules (we refer to the treatment of [BGS] for generalities on Koszul rings, in a much more general context).

When R is a commutative algebra "arising from algebraic geometry", e.g., $R_E = \bigoplus_i H^0(X, E^{\otimes i})$, where X is a projective variety and E some line bundle on X, the Koszul conditions have a convenient interpretation in terms of line bundles due to Lazarsfeld. To see this, it is useful to set the following notation: if F is a sheaf on X, M_F will denote the kernel of the evaluation map $H^0(X, F) \otimes \mathcal{O}_X \to F$. Note that if F is globally generated and locally free on X then M_F is locally free. However, if H is locally free then $H^0(M_F \otimes H)$ is the kernel of the multiplication map $H^0(F) \otimes H^0(H) \to H^0(F \otimes H)$. Therefore, as it is immediate to see, $R_E^{(1)} = \bigoplus_i H^0(X, M_E \otimes E^{\otimes i})$, $R_E^{(2)} = \bigoplus_i H^0(X, M_{M_E \otimes E} \otimes E^{\otimes i})$ and so on. Inductively, let us set $M_E^0 := E$, $M_E^1 := M_E \otimes E$, $M_E^2 := M_{M_E^1} \otimes E, \ldots, M_E^p := M_{M_E^{p-1}} \otimes E$

Received December 12, 1995
1991 Mathematics Subject Classification. Primary 14J10, 14J25.