BALANCED INCOMPLETE BLOCK DESIGNS AND ABELIAN DIFFERENCE SETS[1]

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1. Balanced incomplete block designs

A balanced incomplete block design (b.i.b.) is an arrangement of v objects into b sets of k elements each called blocks such that every object occurs exactly r times and every pair of objects occurs exactly λ times in a block. Counting objects and pairs of objects in two different ways, one obtains the well-known equations

$$bk = rv, \qquad \lambda(v-1) = r(k-1). \tag{1}$$

In this paper we shall first give a new representation of balanced incomplete block designs in terms of permutation matrices, from which several known theorems follow easily. The main body of the paper is concerned with the study of Abelian difference sets. The paper contains several new results together with a self-contained exposition of known results. Theorem 3 gives a new condition for the existence of these difference sets, valid for every prime divisor of $n = k - \lambda$. Theorem 5 generalizes results previously obtained for cyclic difference sets to Abelian difference sets. Theorems 6 and 7 give a new result for cyclic difference sets which disposes of 9 of the 12 cases which are mentioned as unsettled in [3]. Another of these unsettled cases is disposed of by Theorem 3.

For any given b.i.b., we define a $v \times b$ matrix $A = (a_{ij})$ where

$$a_{ij} = 1 \quad \text{if the } i^{\text{th}} \text{ object occurs in the } j^{\text{th}} \text{ block,}$$
$$a_{ij} = 0 \quad \text{otherwise.}$$

From the properties of the design we then have

$$AA^t = (r - \lambda)I + \lambda T, \tag{2}$$

where I is a $v \times v$ unit matrix and T is a $v \times v$ matrix all of whose entries are unity. Since for every real matrix A

$$\text{rank } AA^t = \text{rank } A,$$

it follows that

$$b \geqq v,$$

an inequality first derived by R. A. Fisher [2].

We now assume $b = v$; hence $r = k$ and $\lambda(v-1) = k(k-1)$. Putting

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