RESIDUAL PROPERTIES OF POLYCYCLIC GROUPS

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1. Let P be a group property. A group G is *residually-P* if to any non-identity element in G there is a normal subgroup K of G excluding x, and such that G/K has P. K. A. Hirsch [6] proved that a polycyclic group, which is a soluble group with maximal condition, is residually finite. Our aim is to sharpen this result.

Let π be a set of prime numbers. A *π-number* is a positive integer whose prime divisors lie in π. A *π-group* is a finite group of order a π-number. If π contains just one prime p, we write p-number and p-group. Thus a p-group here always means a finite p-group.

Our main result is

THEOREM 1. *A polycyclic group is residually a π-group for a finite set of primes π.*

We give below an explicit method for constructing the set π. It depends only on the group G and the finite factors occurring in a normal series for G. In particular if G is completely infinite (defined below), we can give a definite bound for π depending only on an invariant of G. A corollary to the theorem is a result of K. W. Gruenberg [3] on finitely generated nilpotent groups, which are a special class of polycyclic groups [7, p. 232].

Our notation is as follows:

If S is a subset of a group G, $\mathrm{Gp}(S)$ = subgroup generated by S;
$G^t = \mathrm{Gp}(g^t \mid g \in G)$, where t is a positive integer;
$[H, K] = \mathrm{Gp}([h, k] = h^{-1}k^{-1}hk \mid h \in H, k \in K)$, where H, K are subsets of G;
$C_G(F)$ = centraliser in G of a factor group F in G;
$\varphi(G)$ = Frattini subgroup of G;
Greek letters are used for sets of primes.
As usual π' is the complementary set to π.

2. For any positive integer t, G^t is a characteristic subgroup of finite index in a polycyclic group G. If t is a π-number, then G/G^t has exponent a π-number and so is a π-group. Hence any normal subgroup H of index a π-number m in G, contains the characteristic subgroup G^m, also of index a π-number. If H is residually a π-group, then for the same reason any $x \in H$ is excluded from some H^n where n is a π-number. Now H^n is normal in G and of index a π-number. These remarks make the following lemma obvious.

LEMMA 1. *If a normal subgroup of index a γ-number in G is residually a δ-group, then G is residually a π-group, where π is the union of γ and δ.*

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