AN EXTENSION OF F. KLEIN'S LEVEL CONCEPT

BY
Klaus Wohlfahrt

Introduction

In the theory of elliptic modular functions F. Klein's concept of the level of a congruence subgroup of the modular group has proved of fundamental importance. That this concept should be extendable to apply, in particular, to arbitrary subgroups of finite index seems never to have been sufficiently appreciated. Such a generalization was, in effect, provided for by R. Fricke, who introduced the notion of the "class" of a modular subgroup, but there is no evidence in his work to suggest that he perceived of it in that light. It is therefore remarkable that, for a congruence subgroup, the identity of the two concepts is a consequence of one of R. Fricke's theorems.

It is the purpose of this note to introduce a general level concept for the modular group and to show its usefulness in investigations into the structure of that group. There will be occasion to throw some light on the connection between certain modular subgroups and Riemann surfaces associated with them. Account will also be taken of some recent results, concerning the structure of the modular group, of I. Reiner and of M. Newman, as well as of old results of R. Fricke.

A theorem of R. Fricke

We take for the modular group ${}_1\Gamma$ the group of all 2×2 matrices with rational integral elements and determinant 1. Any $L \in {}_1\Gamma$ induces a linear transformation of Poincaré's half-plane. If this transformation is different from the identity map and leaves a rational point ζ or ∞ fixed it is of parabolic type and L is called a parabolic matrix. It is well known that such a matrix has the form $\pm A^{-1}U^mA$ where $A \in {}_1\Gamma$ and $\zeta = A^{-1}\infty$ is a parabolic fixed point of the induced transformation or, as we shall say, of L. Here $U = \left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)$ and m is a rational integer, not zero and uniquely determined by L. The modulus $|m|$ of m will be called the amplitude of the parabolic matrix L.

The group consisting of the two matrices $\pm I$, where $I = \left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)$, will be denoted by E. Then let Γ be any subgroup of ${}_1\Gamma$ and, for convenience, let $-I \in \Gamma$, so that the quotient group Γ/E is isomorphic to the group of linear transformations induced by the matrices of Γ. If Γ contains parabolic matrices P their fixed points are called cusps of Γ.

If ζ is a cusp of Γ the subgroup of all matrices of Γ with fixed point ζ is generated by $-I$ and a certain parabolic matrix P. While P is not uniquely determined by Γ and ζ, the choice is only between $\pm P$, $\pm P^{-1}$ of a common amplitude m. This number will be called the amplitude of ζ relative to Γ.

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