APPLICATION OF THE DOMAIN OF ACTION METHOD TO $|xy| \leqq 1$

BY

M. R. VON WOLFF

1. Introduction

In his doctoral thesis, M. Rahman developed what he called the domain of action method in order to answer the question about the closest packing of certain star domains. Before we discuss this method, we need the following definitions:

Let $\mathcal{S}$ be a star domain in the ordinary affine plane, symmetric about O. A set of points $\mathcal{P}$ is said to provide a *packing for* $\mathcal{S}$ if the domains $\{\mathcal{S} + P\}$, where $P \in \mathcal{P}$, have the property that no domain $(\mathcal{S} + P_0)$ contains the center of another in its interior. We shall also say that $\mathcal{P}$ is an *admissible* point set for $\mathcal{S}$.

As definition of the *density* of a point set $\mathcal{P}$ we accept the definition given in [3, p. 5] which is as follows:

Consider the square $|x| < t$, $|y| < t$. Let $A(t)$ denote the number of points of a set $\mathcal{P}$ in the square; then the density of $\mathcal{P}$ (denoted $\mathfrak{D}(\mathcal{P})$) is defined as $\limsup_{t \to \infty} A(t)/4t^2$.

From the definition it follows that for any 2-dimensional lattice $\mathcal{L}$ the density $\mathfrak{D}(\mathcal{L})$ is just the reciprocal of its mesh.

A *norm-distance* is a real-valued function, [1, p. 103], $N(X) = N(OX)$, defined on the plane, such that $N(X)$ is

(1) nonnegative; i.e., $N(X) \geqq 0$;

(2) continuous;

(3) homogeneous; i.e., $N(tX) = |t| N(X)$, where t is any real number.

A convex distance function or *Minkowski distance*, M, is a norm-distance with the additional properties:

(1) $M(PQ) = 0$ implies $P = Q$.

(2) $M(PQ) \leqq M(PR) + M(RQ)$.

Let $\mathcal{P}$ be a point set in the plane and M be a Minkowski distance. The *domain of action* [2, p. 16], $D(P) = D(P, M, \mathcal{P})$ of a point P, relative to M and $\mathcal{P}$, is the set of all points X in the plane for which

$$M(PX) \leqq M(QX) \qquad \text{where } Q \in \mathcal{P}, \quad Q \neq P,$$

when this set is the closure of the set of all points in the plane which are closer to P than any other point of $\mathcal{P}$.

We must note here, however, that the closure of the set of points X such that $M(OX) < M(PX)$ may not always be the same as the set of points X such that $M(OX) \leqq M(PX)$. For there may be a point Y with $M(OY) = M(PY)$ such that for all X in a whole neighborhood of Y, $M(OX) = M(PX)$.

Received April 13, 1962; received in revised form July 22, 1963.