ON KLOOSTERMAN SUMS CONNECTED WITH MODULAR FORMS OF HALF-INTEGRAL DIMENSION

BY

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1. Introduction

In this paper we consider certain exponential sums $W(c, n, \mu, v)$ which are intimately related to the well-known Kloosterman sums and which arise naturally in the theory of modular forms. It is our purpose to show that when the dimension of the modular form is half-integral we can obtain for these sums the asymptotic estimate

$$W(c, n, \mu, v) = O(c^{1/2+\varepsilon}), \qquad \varepsilon > 0, \text{ as } c \to +\infty, \tag{1}$$

where the constant involved depends upon μ and v, but is independent of n (see §2 for definitions and an explanation of the notation).

We use a method of Petersson [4, pp. 16–19] to reduce $W(c, n, \mu, v)$ to a finite sum of sums K_c, for which the estimate (1) has recently been obtained by Malishev [3]. In this way we obtain (1) for all multiplier systems v connected with the modular group and any half-integral dimension. For *integral* dimension the Petersson method alone suffices to derive (1), no use being made in this case of Malishev's result.

The estimate (1) was obtained by Lehmer [1] in the particular case when $W(c, n, \mu, v)$ is the sum connected with $\eta^{-1}(\tau)$, the well-known modular form of dimension $\frac{1}{2}$. It is conceivable that his method could be extended to give the estimate in all the cases for which we obtain it here.

In §5 we remark on the impossibility of obtaining (1) for certain dimensions and choices of the parameters n, μ, v, and conclude with an application of (1) to the estimation of the Fourier coefficients of cusp forms.

2. Preliminaries

Let $\Gamma(1)$ denote the modular group, that is, the set of all 2×2 matrices with rational integral entries and determinant one. Let $\Gamma(n)$ be the principal congruence subgroup of level n, the set of all elements of $\Gamma(1)$ which are congruent, elementwise, to the identity matrix modulo n. If r is a real number, we define a *modular form of dimension* r to be a function $F(\tau)$ meromorphic in the upper half-plane, Im $(\tau) > 0$, such that $\lim_{y\to+\infty} |F(iy)|$ exists (possibly $+\infty$), and satisfying

$$F(M\tau) = v(M)(c\tau + d)^{-r}F(\tau), \tag{2}$$

for each $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(1)$. Here $v(M)$ is complex-valued, independent of

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